The HOD conjecture and its failure

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Introduction

The main question: if the HOD conjecture is false, what does V look like?

The HOD conjecture

Conjecture (Woodin)

Assuming the existence of a supercompact cardinal, it is provable that there is a definable wellorder of Ord^ω*.*

- The conjecture refers to
  - provability in ZFC
  - *boldface* definability over $(V, \in)$
- Connected to inner models for supercompactness, determinacy, forcing, large cardinals beyond choice

The HOD dichotomy

Theorem (Woodin)

If κ is extendible, exactly one of the following holds:

(1) *Every set of ordinals of size at least κ is covered by an ordinal definable set of the same size.*

(2) *Every regular cardinal $\delta \geq \kappa$ is measurable in* HOD.

The existence of a cardinal κ for which (1) holds yields a definable wellorder of Ord^ω, while (2) implies that there is no such wellorder.

Weak extender models

An inner model $N \models \mathrm{ZFC}$ is a *weak extender model of κ is supercompact* if for all $\lambda \geq \kappa$, there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ such that $\mathcal{U} \cap N \in N$ and $P_\kappa(\lambda) \cap N \in \mathcal{U}$.

Theorem (Woodin, Hamkins–Reitz)

If N is a weak extender model of κ is supercompact, every set of ordinals of size at least κ is covered by a set in N of the same size.

Weak extender models absorb all large cardinals above κ.

Optimality of Woodin's HOD dichotomy

Theorem (Woodin)

If κ is extendible, exactly one of the following holds:

- HOD *is a weak extender model of κ is supercompact.*
- *Every regular cardinal $\delta \geq \kappa$ is measurable in* HOD*.*

In particular, the least extendible cardinal is supercompact in HOD and every larger extendible cardinal is extendible in HOD.

Theorem (G.–Poveda)

It is consistent that HOD *is correct about cardinals and cofinalities and the least extendible cardinal is the least supercompact of* HOD*.*

The HOD dichotomy from a strongly compact cardinal

Theorem (G.)

If κ is strongly compact, exactly one of the following holds:

(1) *If $\lambda \geq \kappa$ is singular, λ is singular in* HOD *and $\lambda^{+\text{HOD}} = \lambda^+$.*

(2) *All sufficiently large regular cardinals are measurable in* HOD.

Technical open question: in case (1), is every set of ordinals of size at least κ covered by an ordinal definable set of the same size?

Connection with forcing

A long-standing question of Magidor asks whether it is possible to singularize a cardinal by forcing while preserving a strongly compact cardinal.

Proposition

Suppose $V[G]$ is a ***weakly homogeneous*** *set forcing extension of V and κ is strongly compact in $V[G]$. Then any singular cardinal $\lambda \geq \kappa$ of $V[G]$ is singular in V and $\lambda^{+V} = \lambda^{+V[G]}$.*

Strongly measurable cardinals

An ordinal definable stationary set is η-*unsplittable in* HOD if it admits no ordinal definable partition into η disjoint stationary sets.

Proposition (Woodin)

If a regular cardinal δ has an η-unsplittable stationary subset where $(2^\eta)^{\mathsf{HOD}} < \delta$, then δ is measurable in HOD.

A regular cardinal δ is ω-*strongly measurable in* HOD if $\{\alpha < \delta : \mathrm{cf}(\alpha) = \omega\}$ is η-unsplittable where $(2^\eta)^{\mathsf{HOD}} < \delta$.

In the HOD dichotomies discussed above, "measurable" can be strengthened to "ω-strongly measurable in HOD."

Some ω-strongly measurable cardinals without AC

- (Kleinberg) If $\delta \to (\delta)^{\omega+\omega}$, then δ is ω-strongly measurable.
- (Steel–Woodin) Under AD^+, every regular cardinal $\delta < \Theta$ is ω-strongly measurable.
- (Woodin) If $I_0(\lambda)$ holds (i.e, there is a $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$), then $L(V_{\lambda+1})$ satisfies that every regular $\delta \in (\lambda, \Theta)$ is ω-strongly measurable.
- (Woodin) If there is a Reinhardt cardinal, then all sufficiently large regular cardinals are ω-strongly measurable.

ω-strongly measurable cardinals in ZFC

- By $\mathbb{P}_{\mathrm{max}}$-forcing over $L(\mathbb{R})$, one gets a model of ZFC where ω_1 and ω_2 are ω-strongly measurable in HOD.
- Aksornthong–Gappo–Holland–Sargsyan used $\mathbb{P}_{\mathrm{max}}$-forcing over a Chang-type models to obtain a model of ZFC where ω_1, ω_2, and $\omega_3 = \Theta$ are ω-strongly measurable in HOD.
- Ben-Neria–Hayut built a model of ZFC where all successors of regular cardinals are ω-strongly measurable in HOD from an inaccessible cardinal δ such that $\sup\{o(\kappa) : \kappa < \delta\} = \delta$.
- Assuming $\mathrm{I}_0(\lambda)$, generically well-ordering $V_{\lambda+1}$ produces a ZFC model where a successor of a singular strong limit cardinal is ω-strongly measurable in HOD.

The failure of the HOD conjecture

Theorem (Woodin)

If there is a Reinhardt cardinal and a proper class of supercompact cardinals, then the HOD *conjecture is false.*

The key idea is that one can force the Axiom of Choice under these hypotheses preserving the supercompacts.

We will discuss an improvement of this method due to Usuba using Lowenheim-Skolem cardinals.

I_0 and the "right V"

$\mathsf{I}_0(\lambda)$: there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

Under this hypothesis, one can generalize many structural properties of $L(\mathbb{R})$ to $L(V_{\lambda+1})$.

Certain questions about $L(V_{\lambda+1})$, however, are sensitive to the structure of V_λ, which can be altered by forcing while preserving I_0.

Partition properties

Theorem (Blue–G.)

Assume $\mathsf{I}_0(\lambda)$. *In* $L(V_{\lambda+1})$, *suppose Dependent Choice holds and* $\lambda^+ \to (\lambda^+)^\gamma$ *for all* $\gamma < \lambda$. *Then the* HOD *conjecture is false.*

The proof uses inverse Cramer's limit reflection theorem, Woodin's choice-forcing poset, and determinacy of ordinal games.

Large cardinals in HOD

Assuming the HOD conjecture, all large cardinals above the first extendible are downwards absolute to HOD.

If the HOD conjecture fails in the presence of a supercompact, are there very large cardinals in HOD?

We know there is a proper class of measurable cardinals, and these have fairly high Mitchell order.

Embarrassing open question: does HOD contain a measurable cardinal κ of Mitchell order κ?

Supercompactness in HOD

Theorem (G.)

If κ is strongly compact, $\delta > \kappa$ is ω-strongly measurable in HOD, *and* $\text{cf}(\delta^{+\text{HOD}}) > \omega$, *then* $\text{HOD} \vDash$ "δ *is* $\delta^{+\text{HOD}}$*-supercompact*".

This generalizes, e.g., to $\delta^{+(\omega+1)}$-supercompactness.

The proof involves the following ZFC result:

Theorem (Casey–G.)

If $\text{HOD} \vDash \delta$ is regular, *then* $\text{cf}(\delta^{+\text{HOD}}) \in \{\omega, \text{cf}(\delta), |\delta|, \delta^+\}$.

Reinhardt cardinals and periodicity

- $X \leq^* Y$ if there is a surjection from Y to X
- $\aleph^*(X) = \sup\{\alpha + 1 : \alpha \leq^* X\}$
- γ is a *strong limit cardinal* if for all $\eta < \gamma$, $\gamma \not\leq^* P(\eta)$
- $\aleph^*(V_\omega) = \aleph_1$
- $\aleph^*(V_{\omega+1}) = \Theta$ is a strong limit cardinal under AD
- $\aleph^*(V_{\omega+2}) = \Theta^+$ under AD
- AD apparently cannot tell us anything about $\aleph^*(V_{\omega+3})$

Theorem (G.)

If there is a Reinhardt cardinal, then for all sufficiently large α, $\aleph^*(V_\alpha)$ is a strong limit cardinal if and only if α is odd.

A strong theory

Show that the following are inconsistent with ZF + DC:

- For every $\alpha \geq \omega$, $\aleph^*(V_\alpha)$ is a strong partition cardinal.
- For every even $\alpha \geq \omega$,
  - If $A \subseteq V_{\alpha+1}$, either $A \leq^* V_\alpha$ or $V_{\alpha+1} \leq A$.
  - If $A, B \subseteq 2^{V_\alpha}$, $A \leq_W B$ or $B \leq_W \neg A$.
- There is a Berkeley cardinal.

Thanks