HOD Analysis for Admissible Structures

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This is joint work with Farmer Schlutzenberg.

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Let $x \in \mathbb{R}$ be fixed such that $M_1^\sharp \in L[x]$ and let κ_x be the least inaccessible cardinal of $L[x]$. Let $G \subset \mathrm{Col}(\omega, < \kappa_x)$ be $L[x]$-generic.

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Definition

Let $(\mathcal{F}, \to)$ be the directed system of all Σ-iterates of M_1 which are countable in $L[x, G]$, where $N \to P$ if P is a Σ-iterate of N.

Definition

Let $\mathcal{M}_\infty$ be the direct limit of $(\mathcal{F}, \to)$ by the iteration maps given by Σ.
Let δ_∞ be the Woodin cardinal of $\mathcal{M}_\infty$ and κ_∞ the least inaccessible of $\mathcal{M}_\infty$ greater than δ_∞.
Let Λ be the restriction of Σ to finite stacks of normal trees on $\mathcal{M}_\infty$ such that $\vec{\mathcal{T}} \in \mathcal{M}_\infty | \kappa_\infty$.

Theorem (Steel, Woodin)

Suppose that every real has a sharp and $M_1^\sharp$ is $(\omega, \omega_1, \omega_1)$-iterable. Then $\mathrm{HOD}^{L[x,G]} = \mathcal{M}_\infty[\Lambda]$.

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A naive idea of showing that $\mathcal{M}_\infty \subset \mathrm{HOD}^{L[x,G]}$ would be to try to show that $(\mathcal{F}, \to) \in L[x, G]$ and then compute $\mathcal{M}_\infty$ from this. However: If $i_\mathcal{T} : M_1 \to P$ is in $L[x, G]$, where $\mathcal{T}$ is the canonical tree which makes x generic over M_1 for the extender algebra of M_1, then $i_\mathcal{T}$ singularizes the ω_1 of $L[x, G]$.

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with maps

$$\pi_{(P,s)(N,t)} : H_s^N \to H_t^P,$$

with direct limit $M'_\infty \subset \mathrm{HOD}^{L[x,G]}$.

In order to show that $M_\infty = M'_\infty$ it is crucial that the maps $\pi_{(P,s)(N,t)}$ in $\mathcal{D}$ sufficiently agree with the maps i_{NP} given by Σ if $N, P \in \mathcal{F}$.

In order to show that $M_\infty = M'_\infty$ it is crucial that the maps $\pi_{(P,s)(N,t)}$ in $\mathcal{D}$ sufficiently agree with the maps i_{NP} given by Σ if $N, P \in \mathcal{F}$. This is assured by the Silver indiscernibles of $L[x, G]$ which essentially give fixed points for the embeddings in $(\mathcal{F}, \to)$.

Question

Can one do a similar analysis for admissible structures?

Replacing M_1 with:

Definition

Let $\mathcal{M}_1^{\text{ad}}$ be the least mouse which models

- KP,
- $V = L[E]$,
- there is a Woodin cardinal δ, and
- there is an inaccessible κ such that $\delta < \kappa$ and κ^+ exists.

Let Σ_1^{ad} be the unique $(\omega, \omega_1, \omega_1)$-iteration strategy for $\mathcal{M}_1^{\text{ad}}$.

Replacing $L[x, G]$ with:

Definition

Let $x_1 \in \mathbb{R}$ be such that $x_1 \leq_T \mathcal{M}_1^{\text{ad}}$ and let α_1 be the least β such that

$$L_\beta[x_1] \models \text{KP} + \exists\kappa(\text{“}\kappa \text{ is inaccessible”} \wedge \text{“}\kappa^+ \text{ exists”}).$$

Let $G_1 \subset \text{Col}(\omega, < \kappa_1)$ be $L_{\alpha_1}[x_1]$-generic, where κ_1 is the inaccessible of $L_{\alpha_1}[x_1]$.

Replacing $\mathrm{HOD}^{L[x,G]}$ with:

Definition

Let Σ_1-OD be the set of all elements of $L_{\alpha_1}[x_1, G_1]$ which are ordinal-definable over $L_{\alpha_1}[x_1, G_1]$ via a Σ_1-formula.
Let Σ_1-HOD be the set of all $a \in L_{\alpha_1}[x_1, G_1]$ such that $\mathrm{tc}(a) \subset \Sigma_1$-OD.

Definition

Let $(\mathcal{F}^{\mathrm{ad}}, \to)$ be the directed system of all Σ^{ad}-iterates which are countable in $L_{\alpha_1}[x_1, G_1]$, where $N \to P$ if P is a Σ^{ad}-iterate of N.

Definition

Let $\mathcal{M}_\infty$ be the direct limit of $(\mathcal{F}^{\mathrm{ad}}, \to)$ by the iteration maps given by Σ^{ad}.
Let δ_∞ be the Woodin cardinal of $\mathcal{M}_\infty$ and κ_∞ the least inaccessible of $\mathcal{M}_\infty$ greater than δ_∞.
Let Λ^{ad} be the restriction of Σ^{ad} to finite stacks of normal trees on $\mathcal{M}_\infty$ such that $\vec{\mathcal{T}} \in \mathcal{M}_\infty | \kappa_\infty$.

Theorem (K., Schlutzenberg)

Suppose that there is a Woodin cardinal. Then Σ_1*-HOD* $= \mathcal{M}_\infty[\Lambda^{ad}]$.

The main obstacle in adopting the proof of the analysis of $\mathrm{HOD}^{L[x,G]}$ is finding a suitable replacement for the use of indiscernibles.

Let $\mathbb{R}^+ = \bigcup_{\beta<\kappa} \mathbb{R}^{L_{\alpha_1}[x_1, G_1\restriction\beta]}$.

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Definition

Let T be the tree which searches for a sequence $\langle \gamma_n, \beta_n, m_n \mid n < \omega \rangle$ such that

1. $\kappa < \beta_n < \gamma_n < \kappa^+$,

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3. there is a Σ_1-elementary embedding $\pi \colon L_{\gamma_n}(\mathbb{R}^+) \to L_{\gamma_{n+1}}(\mathbb{R}^+)$ such that $\pi \restriction (\kappa^+)^{L_{\gamma_n}(\mathbb{R}^+)} = \mathrm{id}$,

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4. if $m_n = 0$, then $\pi(\beta_n) > \beta_{n+1}$, and if $m_n > 0$, then $\pi(\beta_n) = \beta_{n+1}$ and $m_{n+1} = m_n - 1$.

Lemma

Let $b = \langle \gamma_n, \beta_n, m_n \rangle_{n<\omega}$ be the left-most branch of T. Let M_b be the direct limit given by b. Then

- *M_b is ill-founded with $wfc(M_b) = L_{\alpha_1}(\mathbb{R}^+)$,*
- *$\sup\{\gamma_n \mid n < \omega\} = \kappa^+$, and*
- *for every $n < \omega$, $\{(\gamma_n, \beta_n, m_n)\}$ is $\Sigma_1 \wedge \Pi_1$-definable over any x-weasel N.*

Generalizing to Σ_n-KP for $n \geq 1$.

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Let $\mathcal{M}_n^{\mathrm{ad}}$ be the least mouse which models

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Theorem (K., Schlutzenberg)

Suppose that there is a Woodin cardinal. Let $n \geq 1$. Then

1. Σ_n-*HOD* $= M_\infty[\Lambda_n]$,
2. Σ_n-*HOD* $\models \Sigma_n$-*KP* $+ \exists\delta($"δ *is Woodin*"$)$,
3. Σ_n-*HOD is a forcing ground of* $L_{\alpha_n}[x_n, G_n]$.

Thank you!