Hod mice as a bridge between determinacy, forcing axioms and infintary combinatorics

Grigor Sargsyan

Institute of Mathematics
Polish Academy of Sciences

Determinacy, inner models and forcing axioms
June 27
ESI, Vienna

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Theorem (Woodin, Adolf-S.-Trang-Zeman-Wilson)

The following theories are equiconsistent.

1. CH $+$ "*there is an* ω_1*-dense ideal on* ω_1"*.*
2. $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$ $+$ "Θ *is a regular cardinal*"*.*

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4. Hod mouse is an iterable hod premouse.

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2. The goal of the talk is to show that hod mice can be used to prove theorems and state conjectures outside inner model theory.
3. There will be four examples of set theoretic themes that hod mice have something to say about.

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Assume proper class of Woodin cardinals.

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Assume MM^{++} *and a class of Woodin cardinals. Then uB is definable over* H_{ω_3}.

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Remark (Personal struggles)

If no natural extension of MM^{++} *can decide* $\mathrm{cf}(\Theta_{uB})$*, then this would seem like a bad news for* MM^{++}*.*

The $\mathrm{cf}(\Theta_{uB})$ problem

Theorem (Woodin)

In the standard models of MM^{++}, $\Theta_{uB} < \omega_3$. *Thus, in these models,* $\mathrm{cf}(\Theta_{uB}) \in \{\omega_1, \omega_2\}$

Sealing

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Definition

Sealing is the conjunction of the following clauses.

1. There is a class of Woodin cardinals.
2. $L(uB, \mathbb{R}) \models \mathrm{AD}^+$.
3. For all V-generic g and $V[g]$-generic h, there is
$$j : L(uB_g, \mathbb{R}_g) \to L(uB_{g*h}, \mathbb{R}_{g*h})$$
such that for all $A \in uB_g$, $j(A) = A_h$.

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Remark

Sealing is the proper hypothesis for studying $L(uB, \mathbb{R})$, and in particular, the $\mathrm{cf}(\Theta_{uB})$ problem.

Sealing and the $\mathrm{cf}(\Theta_{uB})$ problem

Theorem (Blue-S.-Viale)

Each of the following three theories are consistent.

1. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_1$.
2. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_2$.
3. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_3$.

Remark

Whether $\Theta^{L(\mathbb{R})} > \omega_3$ *is possible is a well-known open problem.*

Sealing$+\mathrm{cf}(\Theta_{uB}) = \omega_1$ made precise

Definition

Sealing$+\mathrm{cf}(\Theta_{uB}) = \omega_1$ is the following theory:

1. Sealing and $\mathrm{cf}(\Theta_{uB}) = \omega_1$.
2. If g is a V-generic preserving ω_1 and
$$j : L(uB, \mathbb{R}) \to L(uB_g, \mathbb{R}_g)$$
is the Sealing embedding then $j[uB]$ is Wadge cofinal in uB_g.

The models: $\mathrm{cf}(\Theta_{uB}) = \omega_1$

Theorem (Blue-S.)

Suppose V is a hod premouse, there is a class of Woodin cardinals and κ is a strong cardinal. Let λ be the least inaccessible cardinal above κ, and let $g \subseteq Coll(\omega, <\lambda)$ be V-generic. Then $V[g] \vDash$ "Sealing$+\mathrm{cf}(\Theta_{uB}) = \omega_1$".

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Remark

The theorem is inspired by an argument of Woodin.

Outline of the proof: $\mathrm{cf}(\Theta_{uB}) = \omega_1$

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3. The argument for this uses the fact that in $V[g]$, $V|\lambda$ is universal (i.e. wins the comparison) with respect to countable fully iterable lbr mice (in $< \omega_1 = \lambda$ steps).

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4. This type of comparisons cannot be done by comparing with background constructions (i.e. via the method developed by Steel in his book).

The models: $\mathrm{cf}(\Theta_{uB}) = \omega_3$

Theorem (Blue-S.-Viale)

Suppose V is a hod mouse, there are class of Woodin cardinals and $\kappa < \lambda$ are the first two strong cardinals. Let

1. *$\mathcal{M}$ be the direct limit of all $< \lambda$-iteratae s of V that are obtained by iterating strictly above κ,*
2. *$g \subseteq Coll(\omega, < \lambda)$ be V-generic,*
3. *$K = L(\mathcal{M}, uB_g, \mathbb{R}_g)$,*
4. *$h \subseteq (\mathbb{P}_{max} * Add(1, \omega_3))^K$ be K-generic.*

Then $K \models AD_{\mathbb{R}}$ and

$$K[g * h] \models \textit{Sealing} + \Theta_{uB} = \omega_3.$$

The models: $\mathrm{cf}(\Theta_{uB}) = \omega_2$

1. Assuming class of Woodin cardinals, countably closed posets do not create new uB sets. So this can be done by forcing with $Coll(\omega_2, \omega_3)$.

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1. Assuming class of Woodin cardinals, countably closed posets do not create new uB sets. So this can be done by forcing with $Coll(\omega_2, \omega_3)$.
2. Alternatively, one can force over K with

$$Coll(\omega_1, \mathbb{R}) * Add(1, \omega_2).$$

   This will force Sealing$+\Theta_{uB} = \omega_2$.

$\mathrm{cf}(\Theta_{uB})$ problem with MM

Conjecture

The theories MM^{++}+Sealing+$\mathrm{cf}(\Theta_{uB}) = \omega_1, \omega_2, \omega_3$ + "any large cardinal" are consistent.

Berkeley cardinals

Definition (Bagaria-Koellner-Woodin)

1. κ is called a λ-Berkeley cardinal if for every transitive *M* of size $< \lambda$ and for every $\alpha < \kappa$ there is an elementary embedding $j : M \to M$ with $crit(j) > \alpha$.

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3. κ is a HOD-Berkeley cardinal if clause 1 above holds for all $M \in$ HOD.

Theorem (Bagaria-Koellner-Woodin, ZFC)

If there is a HOD-Berkeley cardinal then the HOD Conjecture fails.

Berkeley cardinals under AD

Theorem (Solovay)

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Conjecture (with Goldberg)

Assume AD^+*. Then* Θ *is a limit of* Θ*-Berkeley cardinals.*

Berkeley cardinals under AD, the key idea of the proof

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4. Then $\pi_\mu : \mathcal{P} \to \mathcal{Q}$ generates an embedding $k : \mathcal{M}_\infty(\mathcal{P}, \Sigma, \kappa) \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda, \pi_\mu(\kappa))$.

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5. The proof uses Siskind-Schlutzenberg-Steel's full normalization technique.

Berkeley cardinals in ZFC

Corollary

Assume AD in $L(\mathbb{R})$*. Then in* $L(\mathbb{R})^{\mathbb{P}_{max}}$*,* ω_1 *is* ω_3*-Berkeley for every* M *that is ordinal definable from a real.*

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Conjecture

The theory $AD^+ + $ "ω_1 *is an Ord-Berkeley cardinal" is consistent relative to large cardinals.*

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Question

Is it possible to force over a model of AD^+ *to obtain a model of ZFC+"*ω_1 *is HOD-Berkeley"?*

Forcing instances of $\mathrm{MM}^{+}+$ over determinacy

Assume $AD_{\mathbb{R}}$**.**

1. For an ordinal $\gamma < \Theta$, the Nairian Models at γ are the models where $X_\gamma = \cup_{\beta<\gamma}(HOD|\beta)^\omega$.
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   1.1 $N_\gamma^- = L_\gamma(X_\gamma)$,
   1.2 $N_\gamma = L(X_\gamma)$,
   1.3 $N_\gamma^+ = HOD_{X_\gamma}$.
2. We say γ is a stable point if letting $\kappa = \Theta^{N_\gamma^+}$,
   2.1 $N_\gamma^+ \models AD_R$,
   2.2 $\mathcal{P}(\mathbb{R}) \cap N_\gamma^+ = \mathcal{P}(\mathbb{R}) \cap N_\kappa^-$.

Nairian Models

Definition

We say κ is a Solovay cardinal if for every $\nu < \kappa$ there is no OD surjection $f : \mathcal{P}(\nu) \to \kappa$.

Theorem (Steel)

Assume $AD^+ + HPC + V = L(\mathcal{P}(\mathbb{R}))$. Suppose δ is a Solovay cardinal such that there is a largest Solovay cardinal $< \delta$ and δ is a limit of Woodin cardinals in HOD. Then δ is a stable point.

Theorem (Woodin)

In the above scenario, $N_\delta^- \models ZF$.

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4. ξ is either a $< \delta$-strong cardinal or a limit of $< \delta$-strong cardinals,
5. ξ is a limit of Woodin cardinals,
6. no $\xi' \leq \xi$ is a measurable limit of $< \delta$-strong cardinals.

Nairian Models

Theorem (Blue-Larson-S.)

Assume HYPO(η, δ, ξ). Let g $\subseteq$ Coll$(\omega, < \eta)$ be generic and M $= L(uB_g, \mathbb{R}_g)$. Let $\mathcal{H}$ be the direct limit of all $< \eta$-iterates of $V|\eta$ and let $\pi : V \to \mathcal{H}$ be the iteration embedding. Then in M,

1. *$\pi(\xi)$ is a stable point and*
2. *if ξ is an inaccessible limit of $< \delta$-strong cardinal then $N^{-}_{\pi(\xi)} \vDash ZF +$ "ω_1 is supercompact".*

Nairian Models

Theorem

The statement $\exists \eta, \delta, \xi$(*HYPO*$(\eta, \delta, \xi)$ + "ξ *is an inaccessible limit of* $< \delta$*-strong cardinals"*) *is consistency wise weaker than a Woodin cardinal that is a limit of Woodin cardinals.*

Corollary

The theory $AD_{\mathbb{R}}$ + "Θ *is a regular cardinal*" + "ω_1 *is supercompact" is weaker than a Woodin cardinal that is a limit of Woodin cardinals.*

Forcing over Nairian Models

Theorem

Assume HYPO(η, δ, ξ) *and suppose* ξ *is the* $n+1st < \delta$*-strong cardinal. Let* $\pi : V \to \mathcal{H}$ *be as before and let* κ_i *for* $i \leq n$ *be the first* $n+1 < \delta$*-strong cardinals of* $\mathcal{H}$ *(thus,* $\pi(\xi) = \kappa_n$*). Let* $g \subseteq Coll(\omega, <\eta)$ *be generic and set* $M = L(uB_g, \mathbb{R}_g)$*. Then in* M*,*

1. $\kappa_0 = \Theta^{N_\xi}$,
2. *for every* $i \leq n-1$, $\kappa_{i+1} = (\kappa_i^+)^{N_\xi}$,
3. *if* $\mathbb{P} = \mathbb{P}_{max} * Add(1, \kappa_0) * Add(1, \kappa_1) * ... * Add(1, \kappa_n)$ *and* $h \subseteq \mathbb{P}$ *is* N_ξ*-generic then*

   $$N_\xi[h] \vDash ZFC + MM^{++}(c) + \forall i \in [0, n](\neg\square(\omega_{2+i}) + \neg\square_{\omega_{2+i}})$$

Nairian Models: the first challenge

Question

Assume HYPO(η, δ, ξ) *and suppose* ξ *is the* $\omega + 1$*st* $< \delta$*-strong cardinal. Can one force* $\neg\Box_{\aleph_\omega}$ *over* $N_{\pi(\xi)}$*?*

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Nairian Models: the second challenge

Corollary

For each $n < \omega$, the theory $\forall i \in [0, n](\neg\Box(\omega_{2+i}) + \neg\Box_{\omega_{2+i}})$ is weaker than a Woodin cardinal that is a limit of Woodin cardinals.

Question (Challenge)

Assume ZFC and suppose there is a superstrong cardinal. Fix $n \in \omega$. Is there a forcing extension satisfying $\forall i \in [0, n](\neg\Box(\omega_{2+i}) + \neg\Box_{\omega_{2+i}})$?

Nairian Models: the third challenge

Question

Is there a Nairian Model in which ω_1 is Ord-Berkeley?

Forcing over Nairian Models: non-convergence of K^c

It follows from the results of the previous slides and a theorem of Jensen-Schimmerling-Schindler-Steel that:

Corollary

It is not provable in ZFC that the K^c constructions with 2^ω-complete background certificates converge.

Towards more ideals

Theorem (Woodin)

Assume $AD_{\mathbb{R}}$ + "Θ *is regular*" + $V = L(\mathcal{P}(\mathbb{R}))$. *Let* $g \subseteq Coll(\omega_1, \mathbb{R}) * Add(1, \omega_2)$ *be generic. Then*

$$V[g] \models CH + \text{"there is an } \omega_1\text{-dense ideal on } \omega_1\text{"}.$$

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Towards more ideals

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Remark

AD_2 is an abstract determinacy like theory for κ^ω. It was isolated by Steel. See the problem list of IMT 2023, Irvine.

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Continuing with the set up of the previous slide, the following can be established.

Theorem

Let μ be the ω_1-supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$. Then $\pi_\mu(\omega_1) = \lambda$.

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Question

What sort of ideal does μ from the above theorem generate in the $Coll(\omega_1, \mathbb{R}) * Add(1, \omega_2) * Add(1, \omega_3)$-extension of N_λ?