The Comparison Lemma

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Plan

(I) Premice, iteration trees, and iteration strategies.

(II) A general comparison process for short-extender mice. The Dodd-Jensen Lemma and the mouse order.

(III) Comparing iteration strategies. Dodd-Jensen and the mouse pair order.

References:

(1) *The comparison lemma*, J. Steel, APAL 2024, 46 pp.

(2) *A comparison process for mouse pairs*, J. Steel, Lecture Notes in Logic v. 51, (CUP) 2022, 536 pp.

Pure extender premice

An *extender over M* is a system of compatible measures $E = \langle E_a \mid a \in [\nu]^{<\omega} \rangle$ giving rise to an elementary $i^M_E \colon M \to Ult(M, E)$. We have

$$X \in E_a \text{ iff } a \in i_E(X).$$

We let $\kappa_E = \mathrm{crit}(i_E)$ and

$$\lambda_E = i_E(\kappa_E).$$

E is *short* iff $\nu \leq \lambda_E$ (so that E_a concentrates on $[\kappa_E]^{|a|}$.)

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The notions are essentially due to W. Mitchell (1974, 1978).

More detail: A *potential pure extender premouse* is an amenable J-structure

$$M = \langle J_\alpha^{\vec{E}}, \in, \vec{E}, \gamma, F \rangle$$

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If M is a potential pure extender premouse, then $\dot{E}^M$ is a sequence of extenders, and either $\dot{F}^M$ is empty (i.e. M is *passive*), or $\dot{F}^M$ codes a new extender F being added to our sequence by M.

In the active case, F is a short extender over M with support λ_F. We set $\text{lh}(F) = o(M)$.

If M is a proper initial segment of some premouse N, then $F = \dot{E}^N_{\text{lh}(F)}$. It may be that N has subsets of κ_F that are not in M, in which case F is *not* an extender over N! F only measures the subsets of κ_F that got into the model before we added F. (The Baldwin-Mitchell idea.)

To make this work we need a fine structural analysis of the first level of N over which a new subset of κ_F is definable.

The main further requirements on F are

(1) (λ-indexing) $M \models \kappa^+$ exists, and $o(M) = \mathrm{lh}(F) = \lambda_F^{+,M}$. $\dot{F}^M$ is the graph of $i_F^M \restriction (M|\kappa^{+,M})$.

(2) (Coherence) $i_F^M(\dot{E}^M) \restriction o(M) + 1 = (\dot{E}^M)^\frown\langle\emptyset\rangle$.

(3) (Initial segment condition, J-ISC) If G is a whole proper initial segment of F, then G must appear in $\dot{E}^M$. If there is a largest whole proper initial segment, then $\dot{\gamma}^M$ is its index in $\dot{E}^M$. Otherwise, $\dot{\gamma}^M = 0$.

(4) If N is a proper initial segment of M, then N is a potential premouse.

Here an initial segment

$$G = F \restriction \eta =_{df} F \cap ([\eta]^{<\omega} \times M)$$

of F is *whole* iff $\eta = \lambda_G$.

Soundness

The basic fine structural notions apply to potential premice.

$$\rho_1(M) = \text{ least } \alpha \text{ s.t. } \exists r \in M(\mathrm{Th}^M_{\Sigma_1}(\alpha \cup \{r\}) \notin M),$$
$$p_1(M) = \text{ first standard parameter of } M$$
$$= \text{ lex-least descending sequence of ordinals } r$$
$$\text{such that } \mathrm{Th}^M_{\Sigma_1}(\rho_1(M) \cup r) \notin M.$$

We allow $\rho_1(M) = o(M)$ and $p_1(M) = \emptyset$. One can define $\rho_n(M)$ and $p_n(M)$ in a similar way.
Premice are acceptable J-structures, and the key to their fine structure is that if they are sufficiently iterable, then their standard parameters are solid and universal.

Approximate definitions: for $\rho = \rho_1(M)$ and $p = p_1(M)$,

(1) p is *1-solid* iff the function $q \mapsto \mathrm{Th}_1^M(\rho \cup q)$, defined on parameters $q <_{\mathrm{lex}} p$, belongs to M.

(2) p is *1-universal* iff $P(\rho)^M \subseteq \mathrm{cHull}_1^M(\rho \cup p)$.

1-solidity is used to show that if $i\colon M \to N = \mathrm{Ult}_0(M, E)$ is the canonical embedding, then $i(p_1(M)) = p_1(N)$.

Definition

M is *1-solid* iff $p_1(M)$ is 1-solid and $1 - \textit{universal}$. M is *1-sound* iff in addition $M = \mathrm{Hull}_1^M(\rho_1(M) \cup p_1(M))$.

Remark. Suppose M is 1-sound, and $N = \mathrm{Ult}_0(M, E)$ where $\rho_1(M) \leq \kappa_E$; then M is the 1-core of N, and i_E^M is the anticore map. That is, $M = \mathrm{cHull}_1^N(\rho_1(N) \cup p_1(N))$, and i_E^M is the anticollapse.

Definition

A *pure extender premouse* is a potential premouse all of whose proper initial segments are n-sound for all n.

If M is a premouse, then for all κ, $P(\kappa) \cap M \subseteq M|(\kappa^+)^M$. This is a strong, local form of GCH.

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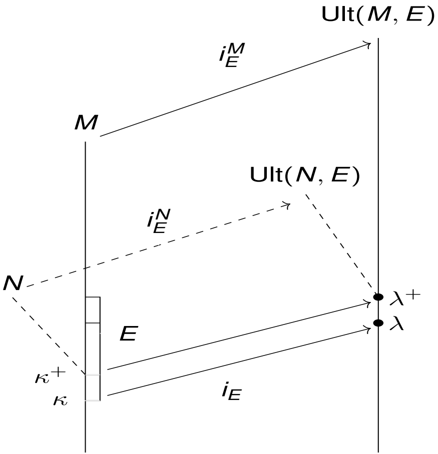
For the mice we construct, solidity and universality are proved by comparison arguments, in an induction that keeps pace with the construction. To obtain soundness, we simply replace the current M by the appropriate hull (core) of M. (I.e. *core down*.) Universality is used to see that we don't lose too much when we core down; for example, we never lose subsets of ω.



M agrees with $\mathrm{Ult}(M, E)$ and $\mathrm{Ult}(N, E)$ to $(\lambda^+)^{\mathrm{Ult}(M,E)}$.

The iteration game

Let M be a premouse. In $\mathcal{G}(M, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $M = \mathcal{M}_0^{\mathcal{T}}$ at the base.

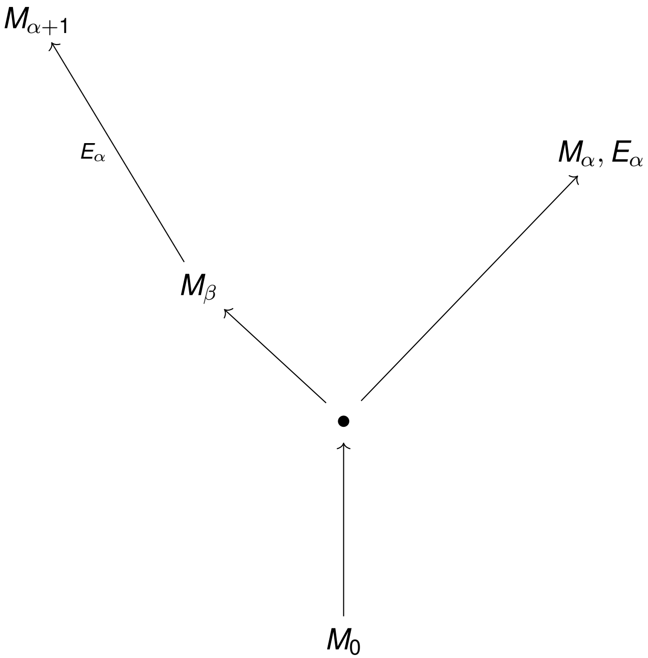
Round $\alpha + 1$*:* I picks an extender $E_\alpha^{\mathcal{T}}$ from the sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ with support $\geq$ the supports of all earlier extenders chosen . Let β be least such that crit$(E_\alpha^{\mathcal{T}}) <$ support $(E_\beta^{\mathcal{T}})$. We set

$$\beta = T\text{-pred}(\alpha + 1),$$

and

$$\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}_k(\mathcal{M}_\beta^{\mathcal{T}} | \eta, E_\alpha^{\mathcal{T}}),$$

where $\langle \eta, k \rangle$ is as large as possible. If E_α is not applied to all of $\mathcal{M}_\beta$, we say that $\mathcal{T}$ *drops* at $\alpha + 1$, and put $\alpha + 1 \in D^{\mathcal{T}}$.
If $\mathcal{M}_{\alpha+1}^{\mathcal{T}}$ is illfounded, then the game is over and I wins.


$M_{\alpha+1}$
E_α
M_α, E_α
M_β
M_0

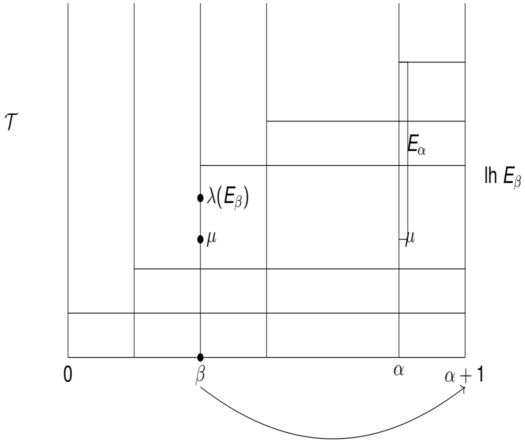
Round λ, for λ limit: II must pick a branch b of $\mathcal{T}$ which is cofinal in λ such that $D^{\mathcal{T}} \cap b$ is finite, and

$$\mathcal{M}_b^{\mathcal{T}} =_{df} \text{ dirlim }_{\alpha \in (b - supD^{\mathcal{T}})} \mathcal{M}_\alpha^{\mathcal{T}}$$

is wellfounded. If he fails to do so, I wins and the game is over. If he succeeds, then we set

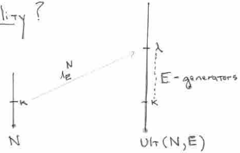
$$\mathcal{M}_\lambda^{\mathcal{T}} = \text{ dirlim }_{\alpha \in (b - \sup D^{\mathcal{T}})} \mathcal{M}_\alpha^{\mathcal{T}}$$

and continue.
If I has not won at some round $\alpha < \theta$, then II wins.
A play of $\mathcal{G}(M, \theta)$ in which II has not yet lost is called a *normal iteration tree* on M.



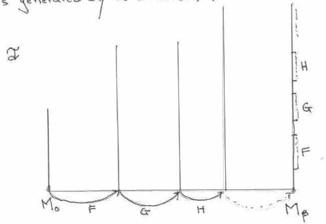
The vertical lines represent the models of $\mathcal{T}$, and the horizontal ones their agreement with one another. β is the $\mathcal{T}$-predecessor of $\alpha + 1$.

Why normality?



$\text{Ult}(N,E) = \{ i(f)(a) \mid f \in N \text{ and } a \in [\lambda]^{<\omega} \}$.

It's generated by $\lambda \cup \text{ran}(i)$.



The individual extenders used going from M_0 to M_β can be recovered from $i^{\mathcal{T}}_{0\beta}$.

e.g. $X \in F_a$ iff $a \in i_F(X)$ iff $a \in i^{\mathcal{T}}_{0\beta}(X)$.

Definition

A *θ-iteration strategy for M* is a winning strategy for II in $\mathcal{G}(M, \theta)$. We say *M* is *θ-iterable* just in case there is such a strategy. If Σ is a strategy for II in $\mathcal{G}(M, \theta)$, and $P = \mathcal{M}_\alpha^\mathcal{T}$ for some $\mathcal{T}$ played by Σ, then we call *P* a *Σ-iterate* of *M*.

Definition

$M \trianglelefteq N$ iff *M* is an initial segment of *N*, 'and $M \triangleleft N$ iff $M \trianglelefteq N$ and $M \neq N$.

Lemma (Comparison, Martin, Mitchell, S. 1985-89)

Let Σ and Γ be $\theta + 1$ iteration strategies for P and Q respectively, where $\theta = \max(|P|, |Q|)^+$; then there are a Σ-iterate R of P and a Γ-iterate S of Q such that either

(a) $R \trianglelefteq S$ and the branch P-to-R does not drop, or

(b) $S \trianglelefteq R$ and the branch Q-to-S does not drop.

Proof sketch

For definiteness, let P and Q be countable, so that Σ and Γ are $\omega_1 + 1$-iteration strategies.
We build padded iteration trees $\mathcal{T}$ on P and $\mathcal{U}$ on Q inductively, by "iterating away the least disagreement" at successor steps, and using our iteration strategies at limit steps. Notation:

$$\mathcal{T}: \quad \text{models } P_\alpha, \text{ extenders } E_\alpha,$$
$$\mathcal{U}: \quad \text{models } Q_\alpha, \text{ extenders } F_\alpha.$$

At step $\alpha + 1$, let

$$\gamma = \text{ least } \beta \text{ such that } P_\alpha | \beta \neq Q_\alpha | \beta.$$

If there is no such β, the comparison terminates.

Otherwise,

$$E_\alpha = \dot{F}_\gamma^{P_\alpha|\gamma}, \text{ and}$$
$$F_\alpha = \dot{F}_\gamma^{Q_\alpha|\gamma}.$$

The rest of step $\alpha + 1$ is determined by the rules for normal, padded iteration trees. (If e.g. $F_\alpha = \emptyset$, then $P_{\alpha+1} = P_\alpha$.)
At limit steps we let $\mathcal{T} \upharpoonright \lambda + 1$ be $\bigcup_{\alpha<\lambda} \mathcal{T} \upharpoonright \alpha$, extended by the branch $\Sigma(\bigcup_{\alpha<\lambda} \mathcal{T} \upharpoonright \alpha)$ if this tree has limit length.
Similarly on the $\mathcal{U}$ side.

Corollary

If P and Q are $\omega_1 + 1$-iterable, then $P|\alpha = Q|\alpha$, where $\alpha = \inf(\omega_1^P, \omega_1^Q)$. That is, their canonical wellorders of the reals by stage-of-construction are compatible.

Proof. If P-to-R does not drop, then $P|\omega_1^P = R|\omega_1^R$. So we can apply the comparison lemma. □

Corollary

If P is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap P$, then x is ordinal definable.

Proof. Let x be the α-th real in P. Then $y = x$ iff y is the α-th real in some $\omega_1 + 1$- iterable premouse. □

Generically absolute definitions

Corollary

Assume AD*. Let $x \in \mathbb{R} \cap P$, where P is an ω_1-iterable countable mouse; then x is Σ^2_1 in a countable ordinal.*

Proof.

Under AD, ω_1 is measurable, so countable ω_1-iterable mice are $\omega_1 + 1$-iterable. If Σ is an ω_1 strategy, then Σ can be coded by set of reals, so "*x* is the α-th real in some ω_1-iterable mouse" is Σ^2_1. □

Corollary

Assume there are arbitrarily large Woodin cardinals, and let P be a countable mouse with an ω_1-iteration strategy that is coded by a Hom_∞ set of reals; then every real is P is $(\Sigma^2_1)^{Hom_\infty}$ is a countable ordinal.

What about the converse–how far do the mice go?

Definition

Mouse Capturing (MC) is the statement: for all reals x and y, if x is $\Sigma_1^2(y)$ in a countable ordinal, then there is an ω_1-iterable y-premouse M such that $x \in M$.

Theorem (Woodin 1990s, Sargsyan 2009)

Assume AD^+*, and suppose there is no boldface pointclass* Γ *such that* $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} +$ *"*θ *is regular"; then Mouse Capturing holds.*

Strategy uniqueness

We don't have a full comparison lemma or a mouse order yet, because how two mice compare might depend on the iteration strategies that are used to compare them. This can happen if the mice have Woodin cardinals. But otherwise, their iteration strategies are unique, and we do have a full comparison, and a mouse order.

Definition

A premouse M is *Q-full* iff M has a largest cardinal, and whenever $M \models \delta$ is Woodin, then $\rho_\omega(M) < \delta$.

Examples:

(1) Any mouse cut at a successor cardinal below its bottom Woodin.

(2) Any active premouse that projects to ω (e.g. $M_n^\sharp$, $M_\omega^\sharp$).

(3) Not M_n or M_ω.

Theorem

Suppose M is sound and Q-full; then M has at most one $|M|^+ + 1$-iteration strategy.

Proof. Suppose Σ and Γ are distinct strategies, and $\Sigma(\mathcal{T}) = b$ and $\Gamma(\mathcal{T}) = c$ where $b \neq c$. Set

$$\delta = \delta(\mathcal{T}) = \sup(\{\text{lh}(E_\alpha^{\mathcal{T}}) \mid \alpha < \text{lh}(\mathcal{T})\}).$$

Lemma (Martin, S., 1985)

δ is Woodin in $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$ with respect to all $f \colon \delta \to \delta$ such that $f \in \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$.

We now compare $\mathcal{T}^\frown b$ with $\mathcal{T}^\frown c$, using Σ and Γ to continue the two trees.

This results in trees $\mathcal{U}$ and $\mathcal{V}$ extending $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ with last models R and S. We may assume wlog that $R \trianglelefteq S$. Then

$$P(\delta)^R \subseteq \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}},$$

so

$$R \models \delta \text{ is Woodin.}$$

So M-to-R has dropped, and R is unsound. Thus $R = S$, S is unsound, and M-to-S drops. Letting $i\colon C \to R$ and $j\colon C \to S$ be the branch embeddings of $\mathcal{U}$ and $\mathcal{V}$, we get $i = j =$ anticore map. This means C-to-R and C-to-S use the same sequence of extenders. This is impossible. □

Corollary

Assume AD^+*, and let* M *be a countable, Q-full,* ω_1*-iterable premouse; then its unique* ω_1*-iteration strategy is coded by a* $\Delta^2_1(\{M\})$ *of reals.*

Stacks of normal trees

Definition

$G(M, \lambda, \theta)$ is the game in which the players play λ rounds, the α-th round being a play of $G(N, \theta)$, where N is the last model of round $\alpha - 1$, or the direct limit along the branch produced by the prior rounds if α is a limit.

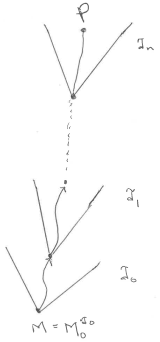
In $G(M, \lambda, \theta)$ I moves at successor stages, by playing an extender or starting a new round if he wishes. If the current round lasts θ moves, then there are no further rounds, and the game is over.
II picks branches at limit stages, and his obligation is just to insure all models are wellfounded, including the direct limit of the base models in the final stack of length λ.

M-stacks

s a stack on M.

$i_s : M \to P$

each $\mathcal{T}_i$ normal



Definition

A (λ, θ)-*iteration strategy* is a M is a winning strategy for II in $G(M, \lambda, \theta)$.

A Q-full M can have at most one $(\lambda, |M|^+ + 1)$-iteration strategy, by the proof for $\lambda = 1$ given above.

Definition

Let M be a premouse; then M is *countably iterable* iff every countable elementary submodel of M is $(\omega_1, \omega_1 + 1)$-iterable.

Countable iterability is what one needs to prove that M is well-behaved in a fine structural sense; for example, that its standard parameter is solid and universal.

Definition

s is an M-stack iff s is a position in some $G(M, \lambda, \theta)$ with I to move that is not yet a loss for II.

Iterates of an iterable structure are iterable, via a *tail strategy*.

Definition (Tail strategy)

Let Ω be a (λ, θ) iteration strategy for M, and s be an M-stack according to Ω with $\mathrm{lh}(s) < \lambda$, and N be the last model of s; then then Ω_s is the $(\lambda - \mathrm{lh}(s), \theta)$ strategy for N

$$\Omega_s(t) = \Omega(s^\frown t).$$

Copying and pullback strategies

Let $\pi \colon M \to N$ be elementary, $\mathcal{T}$ on M with models M_α and extenders E_α. We lift $\mathcal{T}$ to a copied tree $\pi\mathcal{T}$ on N with the same tree order as $\mathcal{T}$, models N_α and extenders F_α. The construction produces elementary copy maps

$$\pi_\alpha \colon M_\alpha \to N_\alpha,$$

such that

(1) if $\beta \leq \alpha$, then $\pi_\alpha \restriction \mathrm{lh}(E_\beta) = \pi_\beta \restriction \mathrm{lh}(E_\beta)$ and $N_\alpha | \mathrm{lh}(F_\beta) = N_\beta | \mathrm{lh}(F_\beta)$, and

(2) if $\beta \leq_T \alpha$, then $\pi_\alpha \circ i^{\mathcal{T}}_{\beta,\alpha} = i^{\pi\mathcal{T}}_{\beta,\alpha} \circ \pi_\beta$.

Set $\pi_0 = \pi$. The successor step is as follows: let $E = E_\alpha$, $\beta = T\text{-pred}(\alpha + 1)$, and

$$F = \pi_\alpha(E),$$
$$P = \mathcal{M}^{*,\mathcal{T}}_{\alpha+1},$$
$$Q = \pi_\beta(P).$$

$F_\alpha = F$, and for $k = k(P)$, let

$$\pi_{\alpha+1} \colon \mathrm{Ult}_k(P, E) \to \mathrm{Ult}_k(Q, F)$$

be the completion of the map

$$\pi_{\alpha+1}([a, f]^{P^k}_E) = [\pi_\alpha(a), \pi_\beta(f)]^{Q^k}_F,$$

for $a \in [\lambda(E)]^{<\omega}$.

Since the copy maps commute with the branch embeddings, at limit steps λ we have a unique elementary $\pi_\lambda \colon M_\lambda \to N_\lambda$ that commutes with the branch embeddings of $\mathcal{T}$ and $\pi\mathcal{T}$ along $[0, \lambda)_T$. It is easy to check (1) and (2).

If $\pi\mathcal{T}$ ever reaches an illfounded model, we stop the construction.

We can copy stacks of plus trees by successively copying the individual plus trees in the stack.

Definition

If Ω is an iteration strategy for N, and $\pi\colon M \to N$ is elementary, then Ω^π is the pullback strategy for M, given by

$$\Omega^\pi(s) = \Omega(\pi s),$$

for all s such that $\pi s \in \mathrm{dom}(\Omega)$.

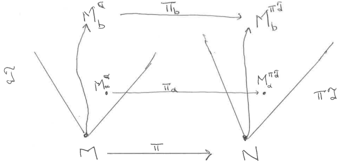
If Ω is a (λ, θ)-iteration strategy for N, then Ω^π is a (λ, θ) iteration strategy for M.

Corollary

Every elementary submodel of a mouse is also a mouse.

Pullback strategies

Given Σ for N, and $\pi : M \to N$



if $b = \Sigma(\pi\mathcal{T})$

then $\Sigma^\pi(\mathcal{T}) = b$

The Dodd-Jensen Lemma

Lemma (Dodd-Jensen)

Let M be Q-full, and $(\theta, \theta + 1)$-iterable, where $\theta = |M|^+$. Let Σ be the unique $(\theta, \theta + 1)$ strategy for M, and s a stack by Σ with last model N, and let

$$\pi \colon M \to P \trianglelefteq N$$

be elementary; then

(1) the branch M-to-N of s does not drop, $P = N$, and

(2) for $i \colon M \to N$ the iteration map, $i(\eta) \le \pi(\eta)$ for all $\eta < o(M)$.

Proof. It is crucial that $(\Sigma_s)^\pi = \Sigma$. This follows from the uniqueness of Σ. □

So the iteration maps on Q-full, iterable mice are pointwise minimal. This implies they are unique.

Corollary

Let M be Q-full, and $(\theta, \theta + 1)$-iterable, where $\theta = |M|^+$. Let Σ be the unique $(\theta, \theta + 1)$ strategy for M, and s and t stacks by Σ with last models P and Q such that $P \trianglelefteq Q$. Then $P = Q$, and if M-to-P does not drop, then M-to-Q does not drop, and the two iteration maps are equal.

Remark There can be distinct non-dropping stacks going from M-to-N. It's just the iteration maps that must be equal.

The mouse order on Q-full mice.

Because it is the most important context for us, we shall assume AD^+ and consider only countable mice. Let M and N be countable, Q-full, (ω_1, ω_1)-iterable premice then

$$M \leq^* N \text{ iff } \exists R, S, \pi(S \text{ is a countable iterate of } N \wedge R \trianglelefteq S \wedge \pi\colon M \to R \text{ is elementary}).$$

Using Dodd-Jensen, one gets that $M \leq^* N$ iff when you

compare them via least disagreement, with last models R on the M side and S on the N side, then M-to-R doesn't drop and $R \trianglelefteq S$. Show $M <^* N$ iff either $R \triangleleft S$ or N-to-S drops.

Mouse limits and HOD

Assume AD^+, and let P be countable, Q-full, and (ω_1, ω_1)-iterable. We define a direct limit system $(\mathcal{F}_P, \prec)$ by

$$Q \in \mathcal{F}_P \text{ iff } Q \text{ is a countable, non-dropping iterate of } P,$$

and for $Q, R \in \mathcal{F}_P$,

$$Q \prec R \text{ iff } R \text{ is a non-dropping iterate of } Q.$$

For $Q \prec R$, let $\pi_{Q,R} \colon Q \to R$ be the (unique) iteration map; then we set

$$M_\infty(P) = \text{dirlim}(\mathcal{F}_P, \prec),$$

where the direct limit is under the $\pi_{Q,R}$ for $Q \prec R$. $(\mathcal{F}_P, \prec)$ is countably directed, so $M_\infty(P)$ is wellfounded, and we then take it to be transitive.

Remarks. Assume AD^+.

(a) $P \equiv^* Q$ iff $M_\infty(P) = M_\infty(Q)$.

(b) $M_\infty(P) \in \text{HOD}$.

(c) $o(M_\infty(P)) <$ boldface δ^2_1.

Definition

P is *full* iff P is Q-full, and whenever $P \leq^* Q$, then $M_\infty(P) \trianglelefteq M_\infty(Q)$.

Theorem (S. 1994)

Assume AD^+*, and suppose Mouse Capturing holds; then*

$$HOD|\delta^2_1 = \bigcup_{P \text{ full}} M_\infty(P).$$

So under these hypotheses, $\text{HOD}|\delta^2_1$ is a premouse, and in particular satisfies GCH.

Mouse pairs

We'd like a full comparison theorem for mice with Woodin cardinals, but their iteration strategies are not unique. We need to compare the strategies too, i.e. to compare *pairs* consisting of a mouse and an iteration strategy for it. This requires that the iteration strategies have certain regularity properties: strong hull condensation, normalizing well, and internal lift consistency.

Strong hull condensation

Roughly, Σ has *strong hull condensation* iff whenever $\mathcal{U}$ is a normal tree on P by Σ, and $\Phi\colon \mathcal{T} \to \mathcal{U}$ is appropriately elementary, then $\mathcal{T}$ is by Σ.

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One must be careful about the elementarity required of Φ, and in particular, the extent to which Φ is required to preserve exit extenders. There are several possible condensation properties here: hull condensation (Sargsyan), strong hull condensation, and still stronger ones.

Strong hull condensation means condensing under *tree embeddings*.

Definition

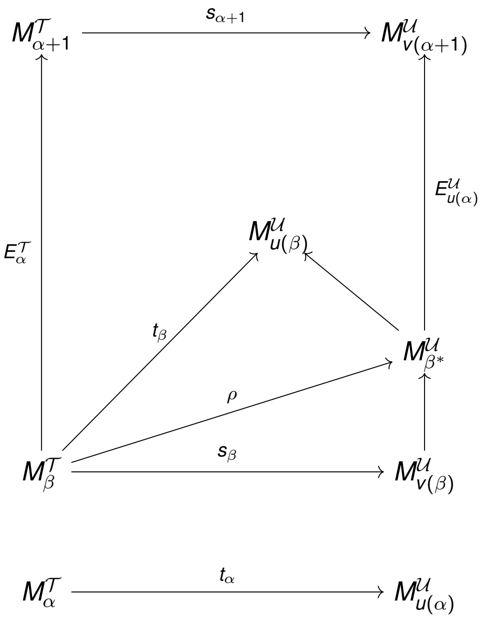
A *tree embedding* of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, v, \langle s_\beta \mid \beta < \mathrm{lh}\mathcal{T}\rangle, \langle t_\beta \mid \beta + 1 < \mathrm{lh}\mathcal{T}\rangle\rangle$$

with various properties, including:

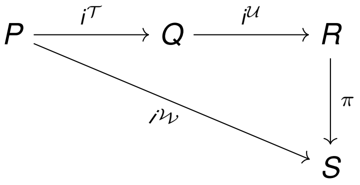
$$E^{\mathcal{U}}_{u(\alpha)} = t_\alpha(E^{\mathcal{T}}_\alpha).$$

The diagram related to successor steps in $\mathcal{T}$ is:



Normalizing well

For $\langle \mathcal{T}, \mathcal{U} \rangle$ a stack on P, we can re-order the use of extenders so as to produce a normal tree $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$, and an embedding of the last model of $\mathcal{U}$ into the last model of $\mathcal{W}$. We call $W(\mathcal{T}, \mathcal{U})$ the *embedding normalization* of the stack $\langle \mathcal{T}, \mathcal{U} \rangle$.



Then Σ *2-normalizes well* iff

$$\langle \mathcal{T}, \mathcal{U} \rangle \text{ is by } \Sigma \text{ iff } W(\mathcal{T}, \mathcal{U}) \text{ is by } \Sigma,$$

and

$$\Sigma^{\pi}_{\langle \mathcal{W} \rangle} = \Sigma_{\langle \mathcal{T}, \mathcal{U} \rangle}.$$

for all such stacks $\langle \mathcal{T}, \mathcal{U} \rangle$. Σ *normalizes well* iff all its tails

Internal lift consistency

Given $M \trianglelefteq N$ and s a stack on N, we can lift s to a stack s^+ on N. (Like copying under the identity map, but the ultrapowers on the N side can use more functions.) A strategy Σ for N is *internally lift consistent* iff whenever $M \trianglelefteq N$ and s^+ is the lift of s on M, then

$$s \text{ is by } \Sigma_M \text{ iff } s^+ \text{ is by } \Sigma.$$

Strategy mice

Definition

A *least branch premouse* (lpm) is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders, and a predicate $\dot{\Sigma}^{\mathcal{M}}$ for an iteration strategy for $\mathcal{M}$.

Remarks

(a) $\mathcal{M}$ has a hierarchy, and a fine structure.

(b) We use Jensen indexing for the extenders in $\dot{E}^{\mathcal{M}}$.

(c) At strategy-active stages in an lpm, we tell $\mathcal{M}$ the value of $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$, where $\mathcal{T}$ is the $\mathcal{M}$-least tree such that $\dot{\Sigma}^{\mathcal{M}}(\mathcal{T})$ is currently undefined.

Mouse pairs

Definition

A *mouse pair* is a pair (P, Σ) such that

(1) P is a countable premouse (pure extender or least branch),

(2) Σ is an iteration strategy defined on all countable stacks on P,

(3) Σ normalizes well, has strong hull condensation, and is internally lift consistent,

(4) if P is an lpm, then Σ is pushforward consistent; i.e. whenever Q is a Σ-iterate of P via s, then $\dot{\Sigma}^{Q} \subseteq \Sigma_s$.

We have limited the notion to countable mice and (ω_1, ω_1)-strategies to smooth some statements later.

Elementary properties of mouse pairs

Definition

$\pi\colon (P,\Sigma) \to (Q,\Psi)$ is *elementary* iff $\pi\colon P \to Q$ is Σ_k elementary, where $k = k(P)$, and $\Sigma = \Psi^{\pi}$.

Lemma

An elementary submodel of a mouse pair is a mouse pair.

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Lemma
An elementary submodel of a mouse pair is a mouse pair.

Definition
(Q,Ψ) is an *iterate of* (P,Σ) iff there is a stack s by Σ with last model Q, and $\Psi = \Sigma_s$.

Lemma

(Iteration maps are elementary) Let (P, Σ) *be a mouse pair, and let s be a stack by* Σ *giving rise to the iteration map* $\pi\colon P \to Q$*; then* $(\Sigma_s)^\pi = \Sigma$.

This property of Σ is called *pullback consistency*.

Lemma

(Dodd-Jensen) The Σ*-iteration map from* (P, Σ) *to* (Q, Ψ) *is the pointwise minimal elementary embedding of* (P, Σ) *into* (Q, Ψ).

Remark. The concept of mouse pair lets us state the Dodd-Jensen in its proper generality.

Comparison

Theorem (Comparison, S. 2015-2021)

Assume AD^+*, and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable; then they have a common iterate* (R, Φ) *such that* R *is countable and at least one of* P*-to-*R *and* Q*-to-*R *does not drop.*

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Definition

(Mouse order) $(P, \Sigma) \leq^* (Q, \Psi)$ iff (P, Σ) embeds elementarily into some iterate of (Q, Ψ).

Corollary

Assume AD^+*; then the mouse order* $\leq^*$ *on mouse pairs of a fixed type is a prewellorder.*

Remark. Again, there is no mouse order on mice with Woodin cardinals.

Mouse pair constructions

Theorem (Woodin, late 1980s)

(AD^+)) *For any Suslin-co-Suslin set B, there is an $(N, \tau, \delta, \Sigma)$ that coarsely captures B.*

This means:

(a) N is countable, $N \models \mathsf{ZFC} +$ "δ is Woodin",

(b) Σ is an iteration strategy for N defined on all $s \in \mathsf{HC}$, and $\Sigma \restriction V_\delta^N \in N$, and

(c) if $i\colon N \to M$ is an iteration map by Σ, and g is $\mathrm{Col}(\omega, i(\delta))$-generic over M, then $i(\tau)_g = B \cap M[g]$.

Inside N, we have the maximal (pure extender, lbr hod) pair construction $\langle (M_{\nu,k}, \Omega_{\nu,k}) \mid \langle \nu, k\rangle \leq_{\text{lex}} \langle \delta, 0\rangle\rangle$:

(a) each $(M_{\nu,k}, \Omega_{\nu,k})$ is a mouse pair,

(b) an E gets added to the sequence of $M_{\nu,0}$ whenever doing so produces a premouse, and E extends to a nice extender E^* in N,

(c) $\Omega_{\nu,k}$ is the strategy for $M_{\nu,k}$ that is induced by Σ,

(d) information about $\Omega_{\nu,k}$ is inserted at strategy-active stages, and

(e) $(M_{\nu,k+1}, \Omega_{\nu,k+1}) = \text{core}(M_{\nu,k}, \Omega_{\nu,k})$.

Comparison arguments show that the construction never breaks down; all levels are mouse pairs whose cores exist, and the E added in (b) is unique.

The main lemma is

Lemma

Assume AD^+*, let* (P, Σ) *be a mouse pair, and let* (N, Ψ) *be a coarse* Γ*-Woodin pair such that* $P \in HC^N$ *and* (N, Ψ) *captures* $Code(\Sigma)$*. Let* $\mathbb{C}$ *be the maximal full background construction of* N *for pairs of the same type; then there is a level* (M, Ω) *of* $\mathbb{C}$ *such that*

(a) (P, Σ) *iterates to* (M, Ω)*, and*

(b) (P, Σ) *iterates strictly past all levels of* $\mathbb{C}$ *that are strictly earlier than* (M, Ω)*.*

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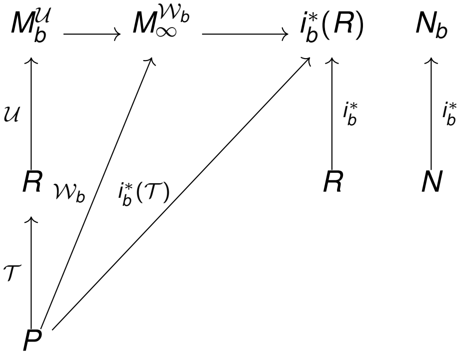
(b) (P, Σ) *iterates strictly past all levels of* $\mathbb{C}$ *that are strictly earlier than* (M, Ω)*.*

Let us sketch why no strategy disagreements show up when we compare (P, Σ) with a level (R, Λ) of $\mathbb{C}$. Let $\mathcal{T}$ from P to R be by Σ. Let $\mathcal{U}$ on R be by both $\Sigma_{\langle \mathcal{T} \rangle}$ and Λ, and let $\Lambda(\mathcal{T}) = b$. We must show $\langle \mathcal{T}, \mathcal{U}^\frown b \rangle$ is by Σ. Let $\mathcal{W}_b = W(\mathcal{T}, \mathcal{U}^\frown b)$ be its embedding normalization.



(i) $\mathcal{T}$ is by Σ, so $i_b^*(\mathcal{T})$ is by Σ.

(ii) There is a tree embedding of $\mathcal{W}_b$ into $i_b^*(\mathcal{T})$, so $\mathcal{W}_b$ is by Σ by strong hull condensation.

(iii) Since Σ normalizes well, $\langle \mathcal{T}, \mathcal{U}^\frown b \rangle$ is by Σ.

Least disagreement comparison

A more effective comparison process yields

Theorem (Sargsyan, S. 2024)

Assume AD^+, *and let* (P, Σ) *and* (Q, Ψ) *be mouse pairs of the same type such that* P *and* Q *are countable and coded by reals* x_P, x_Q. *Let* T_Σ *and* T_Ψ *be Suslin representations of the codesets of the two strategies; then* (P, Σ) *and* (Q, Ψ) *have a common iterate* (R, Φ) *such that* R *is countable in* $L[x_p, x_Q, T_\Sigma, T_\Psi]$.

That the more effective process succeeds relies on results proved using the less effective one.

Hod pair capturing

Least branch hod pairs can be used to compute HOD, provided that there are enough of them.

Definition

(AD^+) *HOD pair capturing* (HPC) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P, Σ) with scope HC such that A is definable over $(\mathrm{HC}, \in, \Sigma)$.

Remarks.

(a) Under AD^+, if (P, Σ) is a mouse pair, then Code(Σ) is Suslin and co-Suslin.

(b) HPC implies that every Suslin-co-Suslin set of reals A is in a symmetric extension of some hod pair (P, Σ). So the theory of $L(A, \mathbb{R})$ is definable over P.

(c) MC implies HPC. We would guess the converse is true, but do not have a proof.

HPC holds in the minimal model of $\mathrm{AD}_{\mathbb{R}} + \theta$ is regular, and somewhat beyond, by Sargsyan's work.

Theorem (Sargsyan, S. 2018)

Assume AD^{+} + $\neg$HPC*; then there is an lbr hod pair* (P, Σ) *such that*
$P \models \mathrm{ZFC} +$ *"there is a Woodin limit of Woodin cardinals".*

Definition

NLE is the statement: there is no ω_1-iteration strategy for a premouse with a long extender on its sequence.

There is no general notion of premice with long extenders yet, but we do have a theory for premice with "not too many" long extenders. NLE says we are in the initial segment of the Wadge hierarchy below the first iteration strategy for such a premouse.

Theorem

Assume AD^+*, and that there is an iterable premouse with a long extender. Let* $\Gamma \subseteq P(\mathbb{R})$ *be such that* $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$ *; then* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*.*

In light of this theorem, the following is almost certainly true:

Conjecture. $(\mathsf{AD}^+ + \mathsf{NLE}) \Rightarrow \mathsf{HPC}$.

HOD as a mouse limit

Definition
(AD$^+$) For (P, Σ) a mouse pair, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

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$M_\infty(P, \Sigma)$ is well-defined by the Dodd-Jensen lemma. Moreover, it is OD from the rank of (P, Σ) in the mouse order. Thus $M_\infty(P, \Sigma) \in \text{HOD}$. It is an initial segment of the lpm hierarchy of HOD *if* (P, Σ) is "full".

Definition
A mouse pair (P, Σ) is full iff for all mouse pairs (Q, Ψ) such that $(P, \Sigma) \leq^* (Q, \Psi)$, we have $M_\infty(P, \Sigma) \trianglelefteq M_\infty(Q, \Psi)$.

Theorem

Assume $\mathrm{AD}_{\mathbb{R}}$ + HPC*; then* $HOD|\theta$ *is the union of all* $M_\infty(P,\Sigma)$ *such that* (P,Σ) *is a full lbr hod pair.*

Theorem

Assume $\mathrm{AD}^+ + V = L(P(\mathbb{R})) + \mathrm{HPC}$*; then* $HOD|\theta$ *is an lpm. Thus* $HOD \models \mathrm{GCH}$.

Suslin representations for mouse pairs

Let (P, Σ) be a mouse pair. A tree $\mathcal{T}$ by Σ is *M_∞-relevant* iff there is a normal $\mathcal{U}$ by Σ extending $\mathcal{T}$ with last model Q such that the branch P-to-Q does not drop. Σ^{rel} is the restriction of Σ to M_∞-relevant trees.
Recall that A is *κ-Suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$.

Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

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Remark. Code(Σ^{rel}) is not α-Suslin, for any $\alpha < |M_\infty(P, \Sigma)|$, by Kunen-Martin. So $|M_\infty(P, \Sigma)|$ is a Suslin cardinal.

Suslin cardinals and mouse limits

Recall that κ is a Suslin cardinal iff there is a set of reals that is κ-Suslin, but not α-Suslin for any $\alpha < \kappa$.

Theorem (Jackson, Sargsyan, S. 2018-2019)

Assume AD^+*. Let* (P, Σ) *be a mouse pair, and let* $\kappa < o(M_\infty(P, \Sigma))$*; then equivalent are*

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Corollary

Assume $\mathsf{AD}^+ + \mathsf{HPC}$*; then equivalent are*

(a) κ *is a Suslin cardinal,*

(b) $\kappa = |\tau|$*, for some cutpoint* τ *of HOD.*

Determinacy models from hod pairs

Woodin limits of Woodins have more strength than one might guess.

Theorem (Sargsyan, S. 2018)

Assume AD^+*, and that there is an lbr hod pair* (P, Σ) *such that* $P \models \mathsf{ZFC}$ + *"*δ *is a Woodin limit of Woodin cardinals + "there are infinitely many Woodin cardinals above* δ*". Then there is a pointclass* Γ *such that*

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Part (1) is due to Sargsyan, and requires weaker hypotheses on *P*. The insight that Woodin limits of Woodins are what you need for (2) is due to Sargsyan.

HOD pairs and Chang models

Relatives of the following theorems were proved earlier by Woodin.

Theorem (Gappo, Sargsyan 2022)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) *such that* P *is countable,* Σ *is coded by a uB set, and* $P \models \mathrm{ZFC}+$ *"there is a Woodin limit of Woodin cardinals"; then the Chang model* $L({}^{\omega}OR)$ *satisfies* AD.

Let $F(\alpha, X)$ iff $X \subseteq P_{\omega_1}({}^{\omega}\alpha)$ and contains a club in $P_{\omega_1}({}^{\omega}\alpha)$.

Corollary (to proof)

Suppose that there are arbitrarily large Woodin cardinals, and that there is an lbr hod pair (P, Σ) such that P is countable, Σ is coded by a uB set, and $P \models \mathrm{ZFC}+$"there is a measurable Woodin cardinal". Let $F(\alpha, X)$ iff X contains a club in $P_{\omega_1}({}^{\omega}\alpha)$; then

(1) $L({}^{\omega}OR)[F] \models \mathrm{AD}_{\mathbb{R}}$, *and*

(2) $L({}^{\omega}OR)[F] \models$ *"for all α, $\{X \mid F(\alpha, X)\}$ is an ultrafilter".*

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(ii) We don't see how to reduce the mouse-existence hypothesis in the corollary to that in the theorem. Both proofs lean heavily of the theory of hod mice, and on the proofs of approximations to HPC that we have now.

(iii) Woodin had already found a proof of the same conclusions from a proper class of Woodin limits of Woodins, using results of Neeman on iterability and long game determinacy at that level.

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Thank you!