Inner models constructed from generalized logics

and their relationship with the standard inner models

Joint work with Kennedy and Magidor, as well as Goldberg, Larson, Rajala, Schindler, Steel, Wilson, and Ya'ar.

Part one of a two part lecture.
The second part will be given by R. Schindler.

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$C(\mathcal{L}^*)$

- **Informal definition:** The inner model $C(\mathcal{L}^*)$ arises from Gödel's L by replacing first order logic $\mathcal{L}_{\omega\omega}$ by an extension $\mathcal{L}^*$ of $\mathcal{L}_{\omega\omega}$.
- Some logics $\mathcal{L}^*$ have absolute syntax: Higher order logics, logics with generalized quantifiers.
- Some logics $\mathcal{L}^*$ have absolute semantics: $L_{\infty\omega}$, $L_{\infty G}$.
- Some have neither: $L_{\kappa\lambda}$.

Possible desirable attributes of inner models

- Forcing absolute.
- Support large cardinals.
- Arise “naturally”.
- Decide questions such as CH.
- Satisfy Axiom of Choice.

- $C(\mathcal{L}_{\omega\omega}) = L$
- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathbb{R})$
- $C(\mathcal{L}_{\omega_1\omega_1}) =$ Chang model, (Chang PSPM 1971)
- $C(\mathcal{L}^2) = \mathrm{HOD}$, (Myhill-Scott PSPM 1971)

Theorem (Gloede 1978, KMV 2021)

Suppose $\mathcal{L}^*$ and its syntax are ZFC-absolute with parameters from L. Then $C(\mathcal{L}^*) = L$.

Corollary

1. $C(\mathcal{L}_A) = L$ *for the smallest admissible set* A *containing* ω. *(Gloede)*
2. $C(\mathcal{L}(Q_\alpha)) = L$ *for all* α.

Definition

Magidor-Malitz quantifier of dimension n:

$$\mathcal{M} \models Q_\alpha^{\mathrm{MM},n} x_1, ..., x_n \varphi(x_1, ..., x_n) \iff$$

$$\exists X \subseteq M(|X| \geq \aleph_\alpha \wedge \forall a_1, ..., a_n \in X : \mathcal{M} \models \varphi(a_1, ..., a_n)).$$

$\mathcal{L}(Q_0^{\mathrm{MM},n})$ is absolute but $\mathcal{L}(Q_1^{\mathrm{MM},2})$ can express Souslinity of a tree.

Consistently, $C(Q_1^{\mathrm{MM},2}) \neq L$, but:

Theorem

If $0^\sharp$ exists, then $C(Q_\alpha^{MM,<\omega}) = L$.

Lemma

Suppose $0^\sharp$ exists and $A \in L$, $A \subseteq [\alpha]^2$. If there is (in V) an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L.

Definition

A logic $\mathcal{L}^*$ is L-tame, if $C(\mathcal{L}^*) = L$.

Can we characterize L-tame logics? Does L-tameness have model theoretic content?

Definition of $C(\mathcal{L}^*)$

For any logic $\mathcal{L}^*$ we define the truth definition and the hierarchy (J'_α), $\alpha \in Lim$, as follows:

$$Tr = \{(\alpha, \varphi(\boldsymbol{a})) : (J'_\alpha, \in, Tr \restriction \alpha) \models \varphi(\boldsymbol{a}), \varphi(\bar{x}) \in \mathcal{L}^*, \boldsymbol{a} \in J'_\alpha, \alpha \in Lim\},$$

where

$$Tr \restriction \alpha = \{(\beta, \psi(\boldsymbol{a})) \in Tr : \beta \in \alpha \cap Lim\},$$

and

$$\begin{array}{lcl} J'_0 & = & \emptyset \\ J'_{\alpha+\omega} & = & \mathrm{rud}_{Tr}(J'_\alpha \cup \{J'_\alpha\}) \\ J'_{\omega\nu} & = & \bigcup_{\alpha<\nu} J'_{\omega\alpha}, \text{ for } \nu \in Lim \\ C(\mathcal{L}^*) & = & \bigcup_{\alpha=\cup\alpha} J'_\alpha. \end{array}$$

Another definition of $C(\mathcal{L}^*)$

$$\begin{cases} L'_0 & = & \emptyset \\ L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \\ L'_{\alpha+1} & = & \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\ C_o(\mathcal{L}^*)^1 & = & \bigcup_\alpha L'_\alpha. \end{cases}$$

For most (but consistently not all) logics the two definitions agree.

Theorem

For any $\mathcal{L}^*$ the classes $C(\mathcal{L}^*)$ and $C_o(\mathcal{L}^*)$ are transitive models of ZF containing all the ordinals. If the syntax of $\mathcal{L}^*$ is KP-absolute, then $C(\mathcal{L}^*) \models AC$.

[1] "o" for "original". Gabe Goldberg pointed out that with the original definition there was a problem with AC.

Definition

A logic $\mathcal{L}^*$ is *adequate to truth in itself* if for all finite vocabularies K there is function $\varphi \mapsto \ulcorner\varphi\urcorner$ from all formulas $\varphi(x_1, \ldots, x_n) \in \mathcal{L}^*$ in the vocabulary K into ω, and a formula $\mathrm{Sat}_{\mathcal{L}^*}(x, y, z)$ in $\mathcal{L}^*$ such that:

1. The function $\varphi \mapsto \ulcorner\varphi\urcorner$ is one to one and has a recursive range.
2. For all admissible sets M, formulas φ of $\mathcal{L}^*$ in the vocabulary K, structures $\mathcal{N} \in M$ in the vocabulary K, and $a_1, \ldots, a_n \in N$ the following conditions are equivalent:
   2.1 $M \models \mathrm{Sat}_{\mathcal{L}^*}(\mathcal{N}, \ulcorner\varphi\urcorner, \langle a_1, \ldots, a_n\rangle)$
   2.2 $\mathcal{N} \models \varphi(a_1, \ldots, a_n)$.

   We may admit ordinal parameters in this definition.

Note: The L-tame logic $\mathcal{L}(Q_0, Q_1, ...)$ is not adequate to truth in itself.

Lemma

If $\mathcal{L}^*$ is adequate to truth in itself, there are formulas $\Phi_{\mathcal{L}^*}(x)$ and $\Psi_{\mathcal{L}^*}(x, y)$ of $\mathcal{L}^*$ in the vocabulary $\{\in\}$ such that if M is an admissible set and $\alpha = M \cap \mathrm{On}$, then for the $C_o(\mathcal{L}^*)$ hierarchy level (L'_α):

1. $\{a \in M : (M, \in) \models \Phi_{\mathcal{L}^*}(a)\} = L'_\alpha \cap M$.
2. $\{(a, b) \in M \times M : (M, \in) \models \Psi_{\mathcal{L}^*}(a, b)\}$ *is a well-order $<'_\alpha$ the field of which is $L'_\alpha \cap M$.*

Theorem

If $\mathcal{L}^*$ is adequate to truth in itself, then $C_o(\mathcal{L}^*)$ satisfies the Axiom of Choice.

We give an example of a logic L^* for which AC may consistently fail in $C_o(L^*)$.

A failure of AC in $C_o(L^*)$

Consider the quantifier

$$M \models Q_n^{ST} xyz\varphi(x,\vec{a})\psi(y,z,\vec{a}) \iff \psi(\cdot,\cdot,\vec{a}) \text{ has order-type } \aleph_{n+1}$$
$$\text{and } \varphi(\cdot,\vec{a}) \text{ is a stationary set of points}$$
$$\text{of cofinality } \aleph_n \text{ in } \psi(\cdot,\cdot,\vec{a}).$$

We let $\mathcal{L}^*$ be the extension of first order logic by the infinitely many quantifiers Q_n^{ST}, $n < \omega$.

Proposition

Relative to the consistency of ZF, it is consistent that the Axiom of Choice fails in the inner model $C_o(\mathcal{L}^*)$ (whence $C_o(\mathcal{L}^*) \neq C(\mathcal{L}^*)$).

Measuring the strengths of logics

- $\mathcal{L}^* \leq \mathcal{L}^+$ if $\mathcal{L}^* \subseteq \mathcal{L}^+$.
- $\mathcal{L}^* \leq' \mathcal{L}^+$ if $C(\mathcal{L}^*) \subseteq C(\mathcal{L}^+)$.
- E.g. $\mathcal{L}(I)$ where I is the Härtig quantifier

$$Ixy\varphi(x,\vec{z})\psi(y,\vec{z}) \iff |\varphi(\cdot,\vec{z})| = |\psi(\cdot,\vec{z})|.$$

  seems quite strong[2] but $C(I)$ ($=_{\text{def}}$ $C(\mathcal{L}(I))$ is relatively weak[3].
- A set theoretic perspective to the strength of logics.
- $C(I)$ is a perfect hit with the extender model approach. (Welch 2022)

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[2] $\Delta(L(I)) = \Delta(\mathcal{L}^2)$, if V=L (V. 1978), where $\Delta(\mathcal{L}^*)$ is the unique smallest extension of $\mathcal{L}^*$ to a logic with the Souslin-Kleene Interpolation property. Even if V=L[E] has no inner model with a Woodin (V. & Welch 2023).

[3] $0^k \notin C(I)$ (Welch 2022). 0^k is the sharp of an inner model of a proper class of measurables.

Recall:

$$\mathrm{HOD} = C(\mathcal{L}^2).$$

NOTE: $\mathcal{L}^2$ is maximal under the inner model order $\leq'$ of logics with finite syntax.

Let

$$\mathrm{HOD}_1 =_{\mathrm{df}} C(\Sigma^1_1).$$

Lemma

1. $C(Q_1^{MM,<\omega}) \subseteq \mathrm{HOD}_1$
2. $C(I) \subseteq \mathrm{HOD}_1$.
3. $C(\mathcal{L}(H)) = \mathrm{HOD}_1$, *where H is the Henkin quantifier*

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x, y, u, v, \vec{z}).$$

Theorem

It is consistent, relative to the consistency of infinitely many weakly compact cardinals that for some λ:

$$\{\kappa < \lambda : \kappa \text{ weakly compact (in } V)\} \notin \mathrm{HOD}_1.$$

Corollary

$C(\Delta(\mathcal{L}^*))$ *need not* *be the same as* $C(\mathcal{L}^*)$.

Proof.

In the above model and with $\mathcal{L}^* = \mathcal{L}(H)$:
$C(\Delta(\mathcal{L}(H))) = C(\Delta(\mathcal{L}^2)) = \mathrm{HOD} \neq \mathrm{HOD}_1 = C(\mathcal{L}(H))$. □

Definition

The cofinality quantifier Q^{cf}_ω is defined as follows:

$$\mathcal{M} \models Q^{\mathrm{cf}}_\omega xy\varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\}$$

is a linear order of cofinality ω.

- Axiomatizable
- Fully compact
- Downward Löwenheim-Skolem down to $\aleph_1$

Definition

$$C^* =_{def} C(\mathcal{L}(Q^{\mathrm{cf}}_\omega))$$

Theorem

If $0^\sharp$ exists, then $0^\sharp \in C^*$. More generally, $x^\sharp \in C^*$ for any $x \in C^*$ such that $x^\sharp$ exists.

Proof.

Let $X = \{\xi < \aleph_\omega : \xi$ is a regular cardinal in L and $\mathrm{cf}(\xi) > \omega\}$. Now $X \in C^*$ and X consists of indiscernibles. Hence

$$0^\sharp = \{\ulcorner\varphi(\vec{x})\urcorner : L_{\aleph_\omega} \models \varphi(\gamma_1, ..., \gamma_n) \text{ for some } \gamma_1 < ... < \gamma_n \text{ in } X\} \in C^*.$$

□

Welch JSL 2022 proves the stronger result $0^k \in C^*$ (assuming 0^k exists). More is known today, see Ralf's lecture today.

It is not known whether there can be a measurable cardinal in C^*

Theorem

- *If there is a measurable cardinal* κ, *then* $V \neq C^*$.
- *If* E *is an infinite set of MC (in* V*), then* $E \notin C^*$.
- *If there is an* L^μ, *then some* L^μ *is contained in* C^*.
- *If there is a Woodin cardinal, then* ω_1 *is (strongly) Mahlo in* C^*. *WC?*
- *Suppose there is a Woodin cardinal* λ. *Then every regular cardinal* κ *such that* $\omega_1 < \kappa < \lambda$ *is weakly compact in* C^*.
- *If there is a proper class of Woodin cardinals, then the regular cardinals* $\geq \aleph_2$ *are indiscernible in* C^*.
- *If* $V = L^\mu$, *then* C^* *is the inner model* $M_{\omega^2}[E]$, *where* $E = \{\kappa_{\omega \cdot n} : n < \omega\}$[4].

[4] This has been generalized to inner models for short sequences of measures. (Ya'ar 2021, Welch 2022)

Theorem

Suppose there is a proper class of Woodin cardinals. Suppose $\mathcal{P}$ is a forcing notion and $G \subseteq \mathcal{P}$ is generic. Then

$$Th((C^*)^V) = Th((C^*)^{V[G]}).$$

Proof.

Let H_1 be generic for $\mathbb{Q}$. Now

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

Let H_2 be generic for $\mathbb{Q}$ over $V[G]$. Then

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$

□

Theorem

- $|\mathcal{P}(\omega) \cap C^*| \leq \aleph_2$.
- *If there are three Woodin cardinals and a measurable cardinal above them, then there is a cone of reals x such that $C^*(x)$ satisfies the Continuum Hypothesis.*
- *It is consistent, relative to the consistency of an inaccessible cardinal, that $V = C^*$ and $2^{\aleph_0} = \aleph_2$.*
- *Open: Do large cardinals imply $C^* \models CH$?*

Stationary logic $\mathcal{L}(\mathrm{aa})$

- $\mathcal{M} \models \mathrm{aa} s \varphi(s, \boldsymbol{a}) \iff \{A \in \mathcal{P}_{\omega_1}(M) \ : \ \mathcal{M} \models \varphi(A, \boldsymbol{a})\}$ contains a club of countable subsets of M.
- $\mathcal{L}(Q_1) \subseteq \mathcal{L}(\mathrm{aa})$.
- $\mathcal{L}(Q_\omega^{cof}) \subseteq \mathcal{L}(\mathrm{aa})$.
- It is not known whether $L(\mathrm{aa})$ is adequate to truth in itself.
- It is not known whether $C_o(aa)$ satisfies AC.

An important property of $C(\mathrm{aa})$: Club Determinacy

- For all α:

$$(J'_\alpha, Tr \restriction \alpha) \models \forall \bar{x}[\mathrm{aa} s \varphi(\bar{x}, \bar{t}, s) \vee \mathrm{aa} s \neg \varphi(\bar{x}, \bar{t}, s)],$$

  where $\varphi(\bar{x}, \bar{t}, s)$ is any formula in $L(\mathrm{aa})$ and $\bar{t}$ is a finite sequence of countable subsets of J'_α.
- CD follows from a proper class of Woodin cardinals[5].
- Technical lemma: Suppose that λ is Woodin and G is $Q_{<\lambda}$ generic over V. If $S \subseteq \lambda$ and $S \in V$ is stationary in V then S is stationary in $V[G]$.
- CD follows from PFA.

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[5]Idea: After some preliminary steps we still have a Woodin cardinal δ and a measurable above. Now we use stationary tower forcing and $j : V \to M$. We compare $C(\mathrm{aa}_\delta)^V$, $C(\mathrm{aa})^M$, level by level, and show, using the below technical lemma, that they are the same model. As we do this, we establish Club Determinacy in V.

Theorem

- *Assuming Club Determinacy, every regular* $\kappa \geq \aleph_1$ *is measurable in* $C(\mathrm{aa})$.
- *Suppose there are a proper class of Woodin cardinals. Then the first order theory of* $C(\mathrm{aa})$ *is (set) forcing absolute*[6].

[6] Suppose $\mathbb{P}$ is a forcing notion and δ a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \to M$ be the (generic) associated elementary embedding. Now $C(\mathrm{aa}) \equiv (C(\mathrm{aa}))^M = C(aa_\delta)$. Let $H \subseteq \mathbb{P}$ be generic over V and $j' : V[H] \to M'$. Again: $(C(\mathrm{aa}))^{V[H]} \equiv (C(\mathrm{aa}))^{M'} = (C(aa_\delta))^{V[H]}$. But $(C(aa_\delta))^{V[H]} = C(aa_\delta)$, since $|\mathbb{P}| < \delta$.

We fix the following notation: $\tau_\xi = \{\mathsf{R}_\in, \mathsf{R}_T, \mathsf{R}_{T^*}\} \cup \{\mathrm{P}_\eta : \eta < \xi\}$, $\tau_\xi^- = \tau_\xi \setminus \{\mathsf{R}_{T^*}\}$. Here $\mathsf{R}_\in$ and R_T are binary and $\mathsf{R}_{T^*}, \mathrm{P}_\eta$ $(\eta < \xi)$, are unary. We use $(P)_\xi$ to denote a sequence $\langle P_\eta : \eta < \xi \rangle$.

An *aa-premouse* is a structure $\boldsymbol{J}_\alpha^T = (J_\alpha^T, \in, T, T^*, (P)_\xi)$ in the vocabulary τ_ξ such that

(1) $T \subseteq \alpha \times (\mathcal{L}(\text{aa}) \times J_\alpha^T)$, and for all $\beta < \alpha$, the set

$$T_\beta = \{\varphi(\boldsymbol{a}) : (\beta, \varphi(\boldsymbol{a})) \in T, \boldsymbol{a} \in J_\beta^T\}$$

is a complete consistent $\mathcal{L}(\text{aa})$-theory in the vocabulary τ_0^- extending the first order theory of $(J_\beta^T, \in, T \restriction \beta)$, where we allow constants c_a for $a \in J_\beta^T$.

(2) T^* is a complete consistent $\mathcal{L}(\text{aa})$-theory in the vocabulary τ_ξ^- extending the first order theory of $(J_\alpha^T, \in, T, (P)_\xi)$ with constants c_a for $a \in J_\alpha^T$.

(3) $\langle P_\eta : \eta < \xi \rangle$ is a continuously increasing sequence of subsets of J_α^T and $\text{aa} s \forall x (\mathrm{P}_\eta(x) \to x \in s) \in T^*$, if $\eta < \xi$.

(4) If $\exists x \varphi(x, \boldsymbol{a}) \in T^*$, then there is $b \in J_\alpha^T$ such that $\varphi(c_b, \boldsymbol{a}) \in T^*$, whenever $\varphi(\vec{x})$ is an $\mathcal{L}(\text{aa})$-formula in the vocabulary τ_ξ^- and $\boldsymbol{a} \in J_\alpha^T$.

(5) The sentence

$$\mathrm{aa}\,\bar{s}\,\exists x\varphi(x,\vec{s},\boldsymbol{a}) \to \mathrm{aa}\,\bar{s}\,\exists x(\varphi(x,\vec{s},\boldsymbol{a}) \wedge \forall y \prec x\neg\varphi(y,\vec{s},\boldsymbol{a}))$$

is in T^*, whenever $\varphi(x,\vec{s},\boldsymbol{y})$ is an $\mathcal{L}(\mathrm{aa})$-formula in the vocabulary τ_ξ^- and $\boldsymbol{a} \in J_\alpha^T$.

(6) The Club Determinacy schema

$$\mathrm{aa}\,\bar{t}(\mathrm{aa}\,s\varphi(\boldsymbol{a},s,\bar{t}\,) \vee \mathrm{aa}\,s\neg\varphi(\boldsymbol{a},s,\bar{t}\,)), \tag{1}$$

where $\varphi(\boldsymbol{a},s,\bar{t})$ is in $\mathcal{L}(\mathrm{aa})$ in the vocabulary τ_ξ^- and $\boldsymbol{a} \in J'_\alpha$, is contained in T^*.

(7) The sentences $\mathrm{aa}\,s\exists x\neg x \in s$ and $\mathrm{aa}\,s(\omega \subseteq s)$ are in T^*.

(8) If $\beta \in \alpha \cap \mathit{Lim}$, $\varphi(\boldsymbol{y})$ is an $\mathcal{L}(\mathrm{aa})$-formula in the vocabulary τ_0^-, $\boldsymbol{b} \in \boldsymbol{J}_\beta^T$, and $\varphi(\boldsymbol{b}) \in T_\beta$, then $\varphi(\boldsymbol{b})^{(J_\beta^T)} \in T^*$.

(9) If $\varphi(s, x, \boldsymbol{y})$ is an $\mathcal{L}(\mathrm{aa})$-formula in the vocabulary τ_ξ^- and $\boldsymbol{a} \in J_\alpha^T$ such that $\mathrm{aa}s\exists x\varphi(s, x, \boldsymbol{a}) \in T^*$, then $\mathrm{aa}s\exists x\varphi(s, x, \boldsymbol{a}) \to \mathrm{aa}s\varphi(s, f_{\varphi(s,x,\boldsymbol{a})}(s), \boldsymbol{a})$ is in T^*. Here we use the term $f_{\varphi(s,x,\boldsymbol{a})}(s)$ to denote the $\prec$-minimal x intuitively satisfying $\varphi(s, x, \boldsymbol{a})$, i.e. we work in a conservative extension of T^*, denoted also T^*, which contains:

$$\mathrm{aa}s\exists x\varphi(s, x, \boldsymbol{a}) \to \mathrm{aa}s(\varphi(s, f_{\varphi(s,x,\boldsymbol{a})}(s), \boldsymbol{a}) \wedge$$

$$\forall z(z \prec f_{\varphi(s,x,\boldsymbol{a})}(s) \to \neg\varphi(s, z, \boldsymbol{a}))).$$

Example

The canonical example of an aa-premouse is

$$\mathcal{N} = (J'_\alpha, \in, Tr \restriction \alpha, Tr_\alpha),$$

where $Tr_\alpha = \{\varphi(\boldsymbol{a}) : (\alpha, \varphi(\boldsymbol{a})) \in Tr$. Note that $\mathcal{N} \in C(\mathtt{aa})$.

Definition

Suppose $\boldsymbol{J}_\alpha^T = (J_\alpha^T, \in, T, T^*, (P)_\xi)$ is an aa-premouse and $\boldsymbol{J}_\beta^S = (J_\beta^S, \in, S, S^*, (P')_{\xi'})$ is an aa-premouse with $\xi \leq \xi'$ and $\alpha \leq \beta$. A mapping $\pi : J_\alpha^T \to J_\beta^S$ is called a *weak elementary embedding* of $\boldsymbol{J}_\alpha^T$ into $\boldsymbol{J}_\beta^S$, in symbols

$$\pi : \boldsymbol{J}_\alpha^T \to \boldsymbol{J}_\beta^S,$$

if π is a first order elementary embedding

$$(J_\alpha^T, \in, T, (P)_\xi) \to (J_\beta^S, \in, S, (P')_{\xi'}) \restriction \tau_\xi^-$$

and for all $\varphi(\bar{x}) \in \mathcal{L}(\mathrm{aa})$ in the vocabulary τ_ξ^- and all $\boldsymbol{a} \in J_\alpha^T$,

$$\varphi(\boldsymbol{a}) \in T^* \iff \varphi(\pi(\boldsymbol{a})) \in S^*.$$

Suppose $(J_\alpha^T, \in, T, T^*, (P)_\xi)$ is an aa-premouse. We define

$$\varphi(s, x, \boldsymbol{a}) \sim \varphi'(s, x, \boldsymbol{a}')$$

if and only if

$$\mathrm{aa}s(f_{\varphi(s,x,\boldsymbol{a})}(s) = f_{\varphi'(s,x,\boldsymbol{a}')}(s)) \in T^*.$$

The aa-ultrapower of $(J_\alpha^T, \in, T, T^*, (P)_\xi)$, has the set M^* of $\sim$-equivalence classes as its domain. The *canonical embedding* $j : J'_\alpha \to M^*$ is defined by $j(a) = [x = a]$. For predicates R we define:

$$R^{M^*}([\varphi_1(s, x, \boldsymbol{a}_1)], \ldots, [\varphi_n(s, x, \boldsymbol{a}_n)]) \iff$$

$$\mathrm{aa}sR(f_{\varphi_1(s,x,\boldsymbol{a}_1)}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a}_n)}(s)) \in T^*.$$

Definition

Let P^* be a new unary predicate symbol and $(P^*)^{M^*} = \{j(a) : a \in J_\alpha^T\}$. We let S^* consist of

$$\psi(P^*, [\varphi_1(s, x, \boldsymbol{a})], \ldots, [\varphi_n(s, x, \boldsymbol{a})]),$$

where $\psi(s, x_1, \ldots, x_n)$ is a τ-formula of $L(\mathrm{aa})$, and

$$\mathrm{aa}\, s\, \psi(s, f_{\varphi_1(s,x,\boldsymbol{a})}(s), \ldots, f_{\varphi_n(s,x,\boldsymbol{a})}(s)) \in T^*.$$

Lemma

The aa-ultrapower $(M, E, S, S^*, (P')_{\xi+1})$, if well-founded, collapses to an aa-premouse $(J_\beta^{\bar{T}}, \in, \bar{T}, \bar{T}^*, (\bar{P})_{\xi+1})$ with vocabulary $\tau_{\xi+1}$. The canonical mapping j, composed with the collapse function $\pi : (M, E) \to (J_\beta^{\bar{T}}, \in)$, is a weak elementary embedding

$$i : (J_\alpha^T, \in, T, T^*, (P)_\xi) \to (J_\beta^{\bar{T}}, \in, \bar{T}, \bar{T}^*, (\bar{P})_{\xi+1}).$$

Lemma

Suppose $(J_\alpha^T, \in, T, T^*)$ is a countable premouse with vocabulary τ and

$$\pi : (J_\alpha^T, \in, T, T^*) \to N$$

is an elementary embedding, where N is an expansion of $(J'_\beta, \in, Tr \restriction \beta, Tr_\beta)$ to a τ-structure. There are $P^+ \subseteq J'_\beta$ and an elementary

$$\pi^* : (M^*, \in^{M^*}, T^{M^*}, S^*, P^*) \to (N, P^+)$$

such that $\pi^*(j(a)) = \pi(a)$ for all $a \in M$.

- We obtain *iterates* $(M_\beta, E_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta)$ of the aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$.
- An aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$ is an **aa-mouse** if its β'th iterate $(M_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta)$ is well-founded for all $\beta < \omega_1$.
- In this case we say that the aa-premouse $(M_0, E_0, T_0, T^*_0, (P)_0)$ is *iterable*.

Proposition

Let $\langle (M_\beta, E_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta), j_{\beta,\gamma} : \beta \leq \gamma \leq \omega_1 \rangle$ *be an aa-iteration of aa-mice. Then for all formulas* $\varphi(\boldsymbol{a})$ *of stationary logic in vocabulary* $\tau^-_{\omega_1}$ *and all* $\boldsymbol{a} \in M_{\omega_1}$*:*

$$\varphi(\boldsymbol{a}) \in T^*_{\omega_1} \iff (M_{\omega_1}, E_{\omega_1}, T_{\omega_1}, (P^{\omega_1})_{\omega_1}) \models \varphi(\boldsymbol{a}).$$

Lemma

Suppose

$$(M_0, \in, T_0, T_0^*, (P)_0) \prec (J'_{\omega\alpha}, \in, Tr \restriction \omega\alpha, Tr_{\omega\alpha}, (P')_0),$$

where α is a limit ordinal and M_0 is countable. Then M_{ω_1} does not have new reals over those in M_0.

Theorem

If Club Determinacy holds, then CH holds in $C(\mathtt{aa})$.

Theorem

If club determinacy holds, there is a Δ^1_3 well-ordering of the reals in $C(\mathtt{aa})$. The reals form a countable Σ^1_3-set.

Proof.

The canonical well-order $\prec$ of $C(\mathtt{aa})$ satisfies:

$$x \prec y \iff \exists z \subseteq \omega(\ z \text{ codes an aa-mouse } M \text{ such that}$$

$$x, y \in M \text{ and } M \models \text{“}x \prec y\text{”}).$$

The right hand side of the equivalence is Σ^1_3 and the claim follows. □

The "plus" version of $C(\mathrm{aa})$

Definition

$\mathcal{M} \models \mathrm{aa}^+ s\varphi(s, \boldsymbol{a}) \iff \{A \in \mathcal{P}_{\omega_1}(M) \ : \ (\mathcal{M}, A)^+ \models \varphi(A, \boldsymbol{a})\}$ contains a club of countable subsets of M.

Example

If (M, E) is a well-founded extensional structure in $C(\mathrm{aa}^+)$, then the transitive collapse of (M, E) is in $C(\mathrm{aa}^+)$. It is not known if the same is true of $C(\mathrm{aa})$.

Proposition (Otto Rajala)

"All" that has been proved for $C(\mathrm{aa})$ holds also for $C(\mathrm{aa}^+)$.

NOTE: The "plus" makes sense also for other models $C(\mathcal{L}^*)$.

Second order logic over countable subsets $\mathcal{L}^2(\omega)$

- $C^2(\omega) =_{\text{def}} C(\mathcal{L}^2(\omega))$
- $\mathcal{L}(Q_\omega^{\text{cof}}) \leq \mathcal{L}^2(\omega)$, so $C^* \subseteq C^2(\omega)$ but consistently[7] $C(\text{aa}) \not\subseteq C^2(\omega)$. Also consistently[8] $C^2(\omega) \not\subseteq C(\text{aa})$.
- $ZFC \vdash C^2(\omega) \subseteq HOD^{C_{\omega_1\omega_1}}$. Hence $Th(C^2(\omega))$ is forcing absolute, assuming a proper class of Woodin limits of Woodins.
- $V = C^2(\omega)$ implies there are no measurable cardinals.
- ω_1^V is strongly Mahlo in $C^2(\omega)$, assuming a Woodin limit of Woodins. WC?

[7] Force over L a Δ^1_3-non constructible real. That real is in $C^2(\omega)$, but the forcing is CCC, so $C(\text{aa}) = L$.

[8] Start with L. Add a Cohen real. Still $C^2(\omega) = L$ as the forcing is homogeneous. Now code by further forcing the Cohen real into stationarity of some stationary sets. The forcing does not add countable sets, so still $C^2(\omega) = L$. But now the Cohen real is in $C(\text{aa})$.

Recent developments

1. Assume a proper class of Woodin cardinals. The reals of $C(\mathrm{aa})$ (and also of C^*) are in M_1. $C(\mathrm{aa})$ has no inner model with a Woodin cardinal. (Magidor-Schindler)
2. Assume Club Determinacy. Then Ultrapower Axiom and GCH hold in $C(\mathrm{aa})$. (Goldberg-Steel)
3. Assume Club Determinacy. If κ is regular in $C(\mathrm{aa})$ with $cof(\kappa) \geq \omega_2^V$, then $o(\kappa)^{C(\mathrm{aa})} \geq 2$. Moreover, then $o(\omega_3^V)^{C(\mathrm{aa})} \geq 3$. (Goldberg-Rajala)
4. If $V = L^\mu$, then $V = C(\mathrm{aa})$. (SQuaRE)
5. There is ongoing investigation on what kind of mice can be found inside models such as C^* and $C(\mathrm{aa}^+)$. (SQuaRE)

Thank you!