Characterizing large cardinals in terms of Löwenheim–Skolem and weak compactness properties of strong logics

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Large cardinals: logical characterizations

Theorem (Magidor)

κ is supercompact iff the LST property holds for $\mathcal{L}^2_{\kappa\omega}$ at κ:
If a sentence of $\mathcal{L}^2_{\kappa\omega}$ has a model, then it has a submodel of size $< \kappa$.

What about weaker large cardinals, e.g., strong cardinals?

Remark

If κ is strong, then the LS property holds for $\mathcal{L}^2_{\kappa\omega}$ at κ:
If a sentence of $\mathcal{L}^2_{\kappa\omega}$ has a model, then it has a model of size $< \kappa$.

But this follows from $V_\kappa \prec_{\Sigma_2} V$, so it has no large cardinal strength.
What about stronger logics, e.g., more infinitary?

Remark

The LS property cannot hold for $\mathcal{L}_{\infty\omega}$ at any cardinal:
This logic can describe any ordinal up to isomorphism.

Strong cardinals and fragments of $\mathcal{L}^2_{\infty\infty}$

To obtain results along the desired lines, we must consider logic fragments not closed under negation.

Definition

$\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ is the fragment of second-order infinitary logic obtained from atomic sentences and their negations by:

- infinitary universal quantifiers (arbitrary length),
- finitary existential quantifiers,
- infinitary disjunctions (arbitrary length), and
- conjunctions of length $< \kappa$.

Theorem (W.)

κ is strong iff the LS property for $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ holds at κ:
If a sentence of $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ has a model, it has one of size $< \kappa$.

Measurable cardinals

We may obtain similar results for measurable cardinals.

Proposition (W.)

The following are equivalent.

- κ is measurable.
- The weak LS property for $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ holds at κ:
  If a sentence of $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ has a model of size κ,
  then it has a model of size $< \kappa$.
- The weak LST property for $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ holds at κ:
  If a sentence of $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ has a model of size κ,
  then it has a submodel of size $< \kappa$.

This is probably not very surprising, though:
measurable cardinals have many different characterizations, and these ones are not particularly elegant.
Perhaps it is more interesting to consider things like Shelah cardinals.

Shelah cardinals

Instead of reflection, consider compactness of the dual logic fragment:

Definition

$\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ is the fragment of second-order infinitary logic obtained from atomic sentences and their negations by:

- infinitary existential quantifiers (arbitrary length),
- finitary universal quantifiers,
- infinitary conjunctions (arbitrary length), and
- disjunctions of length $< \kappa$.

Theorem (Osinski–W.)

κ is Shelah iff weak compactness* holds for $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ at κ:
If a theory of $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ has size κ and every subtheory of size $< \kappa$ has a model of size $< \kappa$, then the theory has a model.

Proof outline

The forward direction (large cardinals $\implies$ reflection/compactness) of these results uses the following lemma.

Lemma

Let $j : V \to M$ be an elementary embedding.
Let $S \in M$ be a structure such that all relations on S are in M.
Let $\varphi \in \mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$.
If $S \models \varphi$ then $M \models S \models j(\varphi)$.

Note that $j(\varphi)$ is not equal to φ if φ has size $> \mathrm{crit}(j)$.

Forward direction of first theorem

If $j : V \to M$ comes from strongness of κ, then from $M \models S \models j(\varphi)$ and elementarity of j, we get $\bar{S} \models \varphi$ for some $\bar{S}$ of size $< \kappa$ as desired.

If $j : V \to M$ comes from measurability and $|S| = \kappa$, this still works.

Proof outline (Shelah cardinals)

- Let κ be a Shelah cardinal.
- We will show weak compactness* for $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ at κ.
- Let $T = \{\varphi_i : i < \kappa\} \subset \mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ be a theory.
- Suppose that for all $\alpha < \kappa$, the subtheory $T_\alpha = \{\varphi_i : i < \alpha\}$ has a model S_α of size $< \kappa$.
- Define $f : \kappa \to \kappa$ by $f(\alpha) = |S_\alpha| + \omega$, so S_α has size $< f(\alpha)$.
- Since κ is Shelah, there is an elementary embedding

$$j : V \to M, \quad \mathrm{crit}(j) = \kappa, \quad V_{j(f)(\kappa)} \subset M.$$

- In M, by elementarity, $\{j(\varphi_i) : i < \kappa\}$ has a model S of size $< j(f)(\kappa)$, which implies that all relations on S are in M.
- For all $i < \kappa$, $M \models S \models j(\varphi_i)$ implies $S \models \varphi_i$ by the lemma.
- Therefore S satisfies the theory T as desired.

Remark on proof of lemma

Recall that we used the following:

Lemma

Let $j : V \to M$ be an elementary embedding.
Let $S \in M$ be a structure such that all relations on S are in M.
Let $\varphi \in \mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$.
If $S \models \varphi$ then $M \models S \models j(\varphi)$.

It follows from a more general fact proved by induction on formulas:

Lemma (easy)

Let $j : V \to M$ be an elementary embedding.
Let $S \in M$ be a structure such that all relations on S are in M.
Let $\varphi(X) \in \mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ and let $f : j(X) \to S \cup \mathrm{Rel}(S)$ be in M.
If $S \models \varphi[f \circ j]$ then $M \models S \models j(\varphi)[f]$.

The only complication here is that j may move the variables X in φ.

Reverse direction: extenders

To obtain strong/Shelah cardinals from reflection/compactness, we use extenders: set-sized objects corresponding to e.e.s $j : V \to M$.

Definition (not quite standard)

Let X and Y be transitive sets.

- $\mathrm{Rel}(X)$ and $\mathrm{Rel}(Y)$ are the sets of all relations on X and Y.
- If $j : V \to M$ is an elementary embedding with $Y \in M$ and $Y \subset j(X)$, the (X, Y)-extender derived from j is the function

$$h : \mathrm{Rel}(X) \to \mathrm{Rel}(Y), \quad h(A) = j(A) \restriction Y.$$

- An (X, Y)-extender means an (X, Y)-extender derived from some such elementary embedding j.
- For $Y \subset X$ there is a trivial extender defined by $h(A) = A \restriction Y$.
- Otherwise, define the critical point of h as the critical point of j.

Large cardinals in terms of extenders

Remark

- κ is strong iff for every $\lambda > \kappa$ there is a (V_κ, V_λ)-extender with critical point κ.
- κ is measurable iff there is a $(V_\kappa, V_{\kappa+1})$-extender with critical point κ. Alternatively: iff there is a $(\kappa, \kappa + 1)$-extender.
- κ is Shelah iff for all $f : \kappa \to \kappa$ there is a $\lambda > \kappa$ and a (V_κ, V_λ) extender h with critical point κ such that $\kappa \in \mathrm{dom}(h(f))$.

Note that since f is a binary relation on κ, it follows that $h(f)$ is a binary relation on λ, but not necessarily a total function $\lambda \to \lambda$.

Remark

To be useful, we will need a more concrete definition of extender as a kind of homomorphism, rather than by quantifying over j.

Extenders as homomorphisms

Definition

An (X, Y)-extender* is a function $h : \mathrm{Rel}(X) \to \mathrm{Rel}(Y)$ such that:

1. h is a Boolean homomorphism on relations of each arity.
2. $h(\{(x_0, x_1, x_2, x_3, x_4) \in X^5 : (x_3, x_1, x_4, x_1) \in A\})$
   $= \{(y_0, y_1, y_2, y_3, y_4) \in Y^5 : (y_3, y_1, y_4, y_1) \in h(A)\}$, etc.
3. $h(\{(x_0, x_1, \ldots, x_4) \in X^5 : \exists z \in x_0\, (z, x_1, \ldots, x_4) \in A\})$
   $= \{(y_0, y_1, \ldots, y_4) \in Y^5 : \exists z \in y_0\, (z, y_1, \ldots, y_4) \in h(A)\}$, etc.

*Actually, unless we add an additional condition this is only a (X, Y) pre-extender, meaning it produces an elementary embedding j from $(V, \in)$ to a possibly illfounded model (M, E) with $Y \in \mathrm{wfp}(M, E)$. For this talk that is enough, since strong, measurable, and Shelah cardinals can be witnessed using pre-extenders.

Proof outline: reverse direction

Suppose the LS property holds for $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ at κ:
If a sentence of $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ has a model, it has one of size $< \kappa$.

- Suppose toward contradiction that κ is not strong.
- F.s. $\lambda > \kappa$, there is no (V_κ, V_λ) extender with critical point κ.
- $(V_\lambda, \in) \models \varphi$ where $\varphi \in \mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ says:
  I am a isomorphic to a rank initial segment of V, and
  there is no (V_κ, me) extender with critical point $\geq \kappa$.
- Note: critical point $\geq \kappa$ means no ordinal less than κ is moved.
- By the LS property, φ has a model of size $< \kappa$.
- This model must be isomorphic to $(V_{\bar{\lambda}}, \in)$ for some $\bar{\lambda} < \kappa$.
- This is a contradiction:
  $(V_{\bar{\lambda}}, \in) \not\models \varphi$ since there is a trivial $(V_\kappa, V_{\bar{\lambda}})$ extender with critical point $\infty \geq \kappa$ defined by $h(A) = A \restriction V_{\bar{\lambda}}$.

Complexity of φ

We claimed that the following statement is $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$:

I am isomorphic to a rank initial segment of V, and
there is no (V_κ, me) extender with critical point $\geq \kappa$.

- "I am a isomorphic to a rank initial segment of V" is $\mathcal{L}^2_{\omega\omega}$.
- Assuming that is the case and considering the transitive collapse:
- "There is no (V_κ, me) extender" is $\forall^\infty \vee^\infty \mathcal{L}^2_{\omega\omega}$.
- Use a second-order variable for each relation on V_κ.
- Use a disjunct for each instance of a relation *among* relations that must be preserved by extenders as homomorphisms, e.g., for each triple $A, B, C \in \text{Rel}(V_\kappa)$ such that $A \cap B = C$.
- φ says that for each variable assignment, at least one condition of the homomorphism definition of "extender" fails.
- To control the critical point, $\forall \alpha < \kappa$ (as a unary relation on V_κ) we use $\mathcal{L}_{\kappa\omega}$ to ensure the value assigned to it in "me" is $\leq \alpha$.

Proof outline: measurable cardinals

Suppose the weak LS property holds for $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ at κ:
If a sentence of $\mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ has a model of size κ,
then it has one of size $< \kappa$.

- Suppose toward contradiction that κ is not measurable.
- There is no $(\kappa, \kappa+1)$ extender with critical point κ.
- $(\kappa+1, \in) \models \varphi$ where $\varphi \in \mathcal{L}^2_{\kappa\omega}(\forall^\infty, \vee^\infty)$ says:
  I am a isomorphic to an ordinal, and
  there is no (κ, me) extender with critical point $\geq \kappa$.
- Since $\kappa + 1$ has size κ, by the weak LS property, φ has a model of size $< \kappa$. (This is why we use κ and $\kappa+1$, not V_κ and $V_{\kappa+1}$.)
- This model must be isomorphic to $(\bar{\lambda}, \in)$ for some $\bar{\lambda} < \kappa$.
- This is a contradiction:
  $(\bar{\lambda}, \in) \not\models \varphi$ since there is a trivial $(\kappa, \bar{\lambda})$ extender with critical point $\infty \geq \kappa$ defined by $h(A) = A \restriction \bar{\lambda}$.

Least strong / least measurable cardinal

If we do not attempt to control the critical point, we do not need $\mathcal{L}_{\kappa\omega}$ and a slightly simpler argument gives the following.

Proposition (W.)

The least strong cardinal is the least cardinal κ such that the LS property holds for $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$ at κ, meaning if a sentence of $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$ has a model, it has one of size $< \kappa$.

In other words, it is the LS number of $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$.

Proposition (W.)

The least measurable cardinal is the least cardinal κ such that the weak LS property holds for $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$ at κ, meaning if a sentence of $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$ has a model of size κ, it has one of size $< \kappa$.

In other words, it is the least cardinal at which we do not get models for any new $\mathcal{L}^2_{\omega\omega}(\forall^\infty, \vee^\infty)$ sentences.

Proof outline: Shelah cardinals

Suppose weak compactness* holds for $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ at κ:
If a theory of $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$ has size κ and every subtheory of size $< \kappa$ has a model of size $< \kappa$, then the theory has a model.

- W.l.o.g. let $f : \kappa \to \kappa$ be strictly increasing and $>$ id.
- For all $i \leq \alpha < \kappa$, $(V_{f(\alpha)+1}, \in, \{\alpha\})$ satisfies a sentence φ_i saying:

  I am a isomorphic to a rank initial segment of V with a distinguished ordinal element of size $\geq i$, and there is a (V_κ, me) extender with critical point $\geq \kappa$ sending f to a partial function whose domain includes that element.

  This is witnessed by the trivial extender (which has crit $= \infty$.)
- Each φ_i is $\mathcal{L}^2_{\kappa\omega}(\exists^\infty, \wedge^\infty)$, so by weak compactness* the theory $\{\varphi_i : i < \kappa\}$ has a model $S \cong (V_\lambda, \in, \{\alpha^*\})$ for some $\alpha^* \geq \kappa$.
- Then there is a (V_κ, V_λ) extender h with critical point $\geq \kappa$ sending f to a partial function whose domain includes $\alpha^* \geq \kappa$.
- h has critical point κ and witnesses κ is Shelah with respect to f.

Thanks for your attention.