Ladder mice

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TU Wien, June 28, 2024

This research funded by the Austrian Science Fund (FWF) [grant Y 1498].

Will work in ZF + AD + $V = L(\mathbb{R})$.

Definition 0.1.

Given inner model M and complexity class Γ, M is Γ-correct if

$$\mathbb{R} \cap M \models \varphi(x) \iff \mathbb{R} \models \varphi(x),$$

for all formulas $\varphi \in \Gamma$ and $x \in \mathbb{R} \cap M$.

Definition 0.2.

For $n < \omega$, $\Delta^1_n(\omega_1)$ denotes the set of reals which are Δ^1_n in a countable ordinal:

$$x \in \Delta^1_n(\omega_1)$$

iff there is $\xi < \omega_1$ and Σ^1_n formulas φ, ψ such that for all $w \in \mathrm{WO}_\xi$, all $n < \omega$,

$$n \in x \iff \varphi(w, n) \iff \neg\psi(w, n).$$

Equivalently, there is a $\xi < \omega_1$ and a Σ^1_n formula φ such that for all $y \in \mathbb{R}$, we have

$$y = x \iff \varphi(w, y).$$

Theorem 0.3.

$\mathbb{R} \cap L_{\omega_1^{ck}} = \Delta^1_1$ (no ordinal parameters).
$L_{\omega_1^{ck}}$ is Σ^1_0-correct but not Σ^1_1-correct.

Theorem 0.4.

$\mathbb{R} \cap L = \Delta^1_2(\omega_1)$.
L is Σ^1_2-correct but not Σ^1_3-correct.

The following results are due to Martin, Mitchell, Steel, Woodin (see [7]):

Theorem 0.5.

$\mathbb{R} \cap M_1 = \Delta^1_3(\omega_1)$.
M_1 is Σ^1_2-correct but not Σ^1_3-correct.

Theorem 0.6.

$\mathbb{R} \cap M_2 = \Delta^1_4(\omega_1)$.
M_2 is Σ^1_4-correct but not Σ^1_5-correct.

Theorem 0.7.

$\mathbb{R} \cap M_3 = \Delta^1_5(\omega_1)$.
M_3 is Σ^1_4-correct but not Σ^1_5-correct.

Etc through M_{2n}, M_{2n+1}.

Theorem 0.8 (Anti-correctness for $L_{\omega_1^{\mathrm{ck}}}, \Pi^1_1$).

Let $M = L_{\omega_1^{\mathrm{ck}}}$. Then we have:

- $(\Pi^1_1)^V \restriction M$ *is* $(\Sigma^1_1)^M$ *(Spector-Gandy)*
- $(\Pi^1_1)^M$ *is* $(\Sigma^1_1)^V \restriction M$, *(Ville?)*

uniformly recursively so: there are recursive $\varphi \mapsto \psi_\varphi$ and $\varphi \mapsto \varrho_\varphi$ such that for all φ which are Π^1_1 formulas, $\psi_\varphi, \varrho_\varphi$ are Σ^1_1 formulas, and for all $x \in \mathbb{R} \cap M$,

$$V \models \varphi(x) \iff M \models \psi_\varphi(x),$$

and

$$M \models \varphi(x) \iff V \models \varrho_\varphi(x).$$

Theorem 0.9.

$$(\Sigma^1_3)^V \restriction (\mathbb{R} \cap L) \text{ is not } L\text{-definable};$$

even

$$(\Sigma^1_3)^V \restriction \omega \text{ is not } L\text{-definable}.$$

In fact, there are Δ^1_3 reals which are not L (e.g. $0^\#$)

Theorem 0.10 (Anti-correctness for M_1, Π^1_3).

We have:

- $(\Pi^1_3)^V \restriction M_1$ *is* $(\Sigma^1_3)^{M_1}$, *(Woodin)*
- $(\Pi^1_3)^{M_1}$ *is* $(\Sigma^1_3)^V \restriction M_1$, *(Martin-Mitchell-Steel)*

uniformly recursively so.

Likewise for M_{2n}, M_{2n+1} and $\Pi^1_{2n+3}, \Sigma^1_{2n+3}$, for all n.

Definition 0.11.

The $\underline{L(\mathbb{R})\text{ language}}$ is language of set theory augmented with a constant $\dot{\mathbb{R}}$ for $\mathbb{R}$.

$\Sigma_n^{\mathcal{J}_\alpha(\mathbb{R})}$ and $\Pi_n^{\mathcal{J}_\alpha(\mathbb{R})}$ always in $L(\mathbb{R})$ language.

Definition 0.12.

For $n \geq 1$:

- $\Sigma_1^{\mathbb{R}}$ denotes Σ_1,
- $\Pi_n^{\mathbb{R}}$ denotes $\neg\Sigma_n^{\mathbb{R}}$, so $\Pi_1^{\mathbb{R}} = \Pi_1$
- $\Sigma_{n+1}^{\mathbb{R}}$ denotes $\exists^{\mathbb{R}}\Pi_n^{\mathbb{R}}$.

Definition 0.13.

Let $\alpha \in \mathrm{OR}$ and $n \geq 1$.

$\mathrm{OD}_{\alpha n}$ denotes the set of $x \in \mathbb{R}$ such that for some $\xi < \omega_1$ and some Σ_n formula φ,

$$y = x \iff \mathcal{J}_\alpha(\mathbb{R}) \models \varphi(w, x)$$

for all $w \in \mathrm{WO}_\xi$ and all $y \in \mathbb{R}$.

Likewise $\mathrm{OD}_{\alpha n}^{\mathbb{R}}$, but with $\Sigma_n^{\mathbb{R}}$ replacing Σ_n.

We will first consider the case $\alpha = 1$, so $\mathcal{J}_\alpha(\mathbb{R}) = \mathcal{J}(\mathbb{R})$.

Remark 0.14.

For $n \geq 1$, $(\Sigma_n^{\mathbb{R}})^{\mathcal{J}(\mathbb{R})}$ is recursively equivalent to $\Sigma_n^{\mathcal{J}(\mathbb{R})}$.

So for $\alpha = 1$ case, will just write "Σ_n" and "Π_n".

Corollary 0.15.

Let $M_{<\omega} = \mathrm{stack}_{n<\omega} M_n^{\#}$.
We have:

- $\mathrm{OD}_{11}^{\mathbb{R}} = \mathrm{OD}_{11} = M_{<\omega} \cap \mathbb{R}$.
- $M_{<\omega}$ *is projectively correct but not* $\Sigma_2^{\mathcal{J}(\mathbb{R})}$*-correct.*

Theorem 0.16 (Woodin, [1]).

Let λ *be a limit ordinal. Then* $\mathrm{OD}_{\lambda 1} = \mathbb{R} \cap M$ *for a mouse* M.

Remark 0.17.

Rudominer [3] showed that $\mathrm{OD}_{\alpha n}^{\mathbb{R}} = \mathbb{R} \cap M$ for certain other (α, n) with $[\alpha, \alpha]$ projective-like: $\alpha \leq \omega_1^{\omega_1}$ and either $\mathrm{cof}(\alpha) > \omega$ or $[\mathrm{cof}(\alpha) \leq \omega$ and $n = 1]$.

Ladder mice:

Definition 0.18.

M-ladder the least mouse M such that there is $\langle\theta_n\rangle_{n<\omega}$ such that:

- θ_n is an M-cardinal,
- $M_n^{\#}(M|\theta_n) \lhd M$ and $M_n^{\#}(M|\theta_n) \models$"$\theta_n$ is Woodin".

Write $M_{\mathrm{ld}} = M$.

Remark 0.19.

- $M_{\mathrm{ld}} \models$"there are no Woodin cardinals".
- $M_{\mathrm{ld}} \not\models \mathrm{ZFC}$.

Theorem 0.20 (Rudominer, Woodin).

$\mathrm{OD}_{12}^{\mathbb{R}} = \mathrm{OD}_{12} = \mathbb{R} \cap M_{\mathrm{ld}}$.

Theorem 0.21 (S., [4]).

Assume ZF + AD + V $=$ L$(\mathbb{R})$. Let α with $[\alpha, \alpha]$ a projective-like gap and either:

- *α is a limit of countable cofinality, or*
- *$\alpha = \beta + 1$ where β does not end a strong gap.*

Then:

- $\mathrm{OD}_{\alpha n} = \mathrm{OD}^{\mathbb{R}}_{\alpha n}$.
- *There is a mouse M such that $\mathrm{OD}_{\alpha n} = \mathbb{R} \cap M$.*

Remark 0.22.

Rudominer [3] proved other instances, e.g. projective-like $[\alpha, \alpha]$ where α has uncountable cofinality.

Recall:

- $[\alpha, \beta]$ is a gap iff this interval is maximal such that $\mathcal{J}_\alpha(\mathbb{R}) \preccurlyeq_1 \mathcal{J}_\beta(\mathbb{R})$.
- A gap $[\alpha, \beta]$ is projective-like iff $\mathcal{J}_\alpha(\mathbb{R}) \not\models$ KP.
- The non-projectve-like gaps are divided into weak and strong.

Theorem 0.23 (?).

Assume ZF + AD + $V = L(\mathbb{R})$, and let $[\alpha, \alpha]$ be as before. Then there is a real x_0 such that for all reals x, there is an (x, x_0)-mouse $M = M_{\mathrm{ld}}^{\alpha}(x, x_0)$ analogous to M_{ld}, and there is $\bar{\alpha}$ and a cofinal Σ_1-elementary

$$\sigma : \mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M) \to \mathcal{J}_{\alpha}(\mathbb{R}),$$

such that:

- $\Pi_2^{\mathcal{J}_{\alpha}(\mathbb{R})}(\{x_0\})$ *is* $\Sigma_2^{\mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M)}(\{x_0\})$*,*
- $\Pi_2^{\mathcal{J}_{\bar{\alpha}}(\mathbb{R}^M)}(\{x_0\})$ *is* $\Sigma_2^{\mathcal{J}_{\alpha}(\mathbb{R})}(\{x_0\})$*,*

uniformly recursively.

Remark 0.24.

(Maybe this follows from DST arguments already? But IMT proof should be new; [4].)

Ladder mice at end of weak/strong gaps (see paper):

Theorem 0.25 (S., [4]).

Let $[\alpha, \beta]$ be a weak gap, or $\beta = \gamma + 1$ where γ ends a strong gap. Then for a cone of reals x, there is an x-mouse $M^{\beta}_{\mathrm{ld}}(x)$ definable from x over $\mathcal{J}_{\beta}(\mathbb{R})$, analogous to M_{ld} over $\mathcal{J}(\mathbb{R})$.

$M^{\beta}_{\mathrm{ld}}(x)$ has infinitely many Woodins; a "ladder" ascends to its least Woodin.

Remark 0.26.

$M^{\beta}_{\mathrm{ld}}(x)$ is defined using earlier work of Steel, S., analysing $\mathcal{J}_{\beta}(\mathbb{R})$ as a derived model [6].

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$$\left(\mathcal{J}(\mathbb{R}) \models \varphi(x)\right) \iff \bigwedge_{n<\omega} \psi_{\varphi,n}(x).$$

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Likewise conversely, $\vec{\psi} \mapsto \varphi_{\vec{\psi}}$.

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Let T_n be the canonical tree projecting to a universal Σ^1_{2n} set $\subseteq \omega \times {}^\omega\omega \times {}^\omega\omega$.

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Let T_n be the canonical tree projecting to a universal Σ^1_{2n} set $\subseteq \omega \times {}^\omega\omega \times {}^\omega\omega$.
For a Π_1 formula $\varphi(u, w)$ and the Σ_2 formula

$$\varrho(u) \iff \exists^{\mathbb{R}} w \, \varphi(u, w),$$

let T_ϱ be the tree building $(x, w, \vec{y})$ such that:

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$$\varrho(u) \iff \exists^{\mathbb{R}} w\ \varphi(u, w),$$

let T_ϱ be the tree building $(x, w, \vec{y})$ such that:

- $x, w \in {}^\omega\omega$,
- $\vec{y} : \omega \to \mathrm{OR}$ is the interlacing of $\langle y_n \rangle_{n<\omega}$ with $y_n : \omega \to \mathrm{OR}$ for each n,
- $(\psi_{\varphi,n}, x, w, y_n) \in [T_n]$ for all n,
- $\vec{y} \restriction n$ involves only entries from $y_0, \ldots, y_{n-1}$.

Then

$$p[T_\varrho] = \{x \mid \mathcal{J}(\mathbb{R}) \models \varrho(x)\}.$$

Fix a Π_1 formula $\varphi(u, w)$ and the Σ_2 formula

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Let N be a projectively correct model. Then

$$T_\varrho^N \text{ denotes } [T_\varrho \text{ as computed in } N].$$

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If N' is projectively correct and $\mathbb{R}^N \subseteq \mathbb{R}^{N'}$, there is a natural embedding

$$\pi_{NN'} : T_\varrho^N \to T_\varrho^{N'}.$$

In particular, $\pi_{NV} : T_\varrho^N \to T_\varrho$, so $p[T_\varrho^N] \subseteq p[T_\varrho]$.

Consider $N = M_{\text{ld}}$. For reals $x \in M_{\text{ld}}$,

$$(T_\varrho^{M_{\text{ld}}})_x \text{ is illfounded} \implies (T_\varrho)_x \text{ is illfounded} \implies \mathcal{J}(\mathbb{R}) \models \varrho(x).$$

But the converse is not clear; maybe there are reals $x \in M_{\text{ld}}$ such that

$$(T_\varrho^{M_{\text{ld}}})_x \text{ is wellfounded}$$

but

$$(T_\varrho)_x \text{ is illfounded, so } \mathcal{J}(\mathbb{R}) \models \varrho(x).$$

Theorem 0.27 (Woodin, [2]).

There is $\gamma < \omega_2^{M_{\text{ld}}}$ such that for all $x \in M_{\text{ld}}$, the following are equivalent:

- $\mathcal{J}(\mathbb{R}) \models \varrho(x)$
- $(T_\varrho)_x$ *is illfounded*
- $(T_\varrho^{M_{\text{ld}}})_x$ *is illfounded or has rank* $\geq \gamma$.

So M_{ld} knows $\Sigma_2^{\mathcal{J}(\mathbb{R})}$ truth. Proof uses stationary tower forcing.

Recall $\mathrm{OD}_{12} = \mathrm{OD}_{12}^{\mathbb{R}}$.

Theorem 0.28 (Rudominer, $\approx$ 2000).

$\mathbb{R} \cap M_{\mathrm{ld}} \subseteq \mathrm{OD}_{12}$.

Theorem 0.29 (Woodin, 2018, [2]).

$\mathrm{OD}_{12} \subseteq \mathbb{R} \cap M_{\mathrm{ld}}$.

Theorem 0.30 (Rudominer, Woodin).

$\mathrm{OD}_{12} = \mathbb{R} \cap M_{\mathrm{ld}}$.

Remark 0.31.

Steel showed that M_{ld} can definably identify the parameter γ.

What about anti-correctness?

$$(M_{\mathrm{ld}}, \Pi_2^{\mathcal{J}(\mathbb{R})})$$

is analogous to $(M_1^{\#}, \Pi_3^1)$ and to $(L_{\omega_1^{\mathrm{ck}}}, \Pi_1^1)$.

Want recursive functions $\varphi \mapsto \psi_\varrho$ and $\varphi \mapsto \varrho_\varphi$ such that for all Π_2 formulas φ, ψ_φ and ϱ_φ are Σ_2 and for all $x \in \mathbb{R} \cap M_{\mathrm{ld}}$,

$$\mathcal{J}(\mathbb{R}) \models \varphi(x) \iff \mathcal{J}(\mathbb{R}^{M_{\mathrm{ld}}}) \models \psi_\varphi(x) \tag{1}$$

and

$$\mathcal{J}(\mathbb{R}^{M_{\mathrm{ld}}}) \models \varphi(x) \iff \mathcal{J}(\mathbb{R}) \models \varrho_\varphi(x). \tag{2}$$

For (2), use Rudominer's earlier work.
For (1), need more.

Recall for M_1, and φ is Π^1_3:

$$\mathbb{R} \models \varphi(x) \iff \mathbb{R}^{M_1} \models \psi_\varphi(x),$$

ψ_φ is Σ^1_3, $\psi_\varphi(x)$ says "there is a Π^1_2-iterable $\varphi(x)$-prewitness".

Definition 0.32.

Consider Π^1_3 formula

$$\varphi(u) \iff \forall z\, \tau(u, z),$$

where τ is Σ^1_2.

Let $x \in \mathbb{R}$. A $\varphi(x)$-prewitness is a pair (N, δ) such that:

(i) N is a premouse,

(ii) $x \in N$,

(iii) $N \models \mathrm{ZF}^- +$"$\delta$ is Woodin",

(iv) $N \models$"it is forced by the extender algebra at δ that $\tau(x, \dot{z})$, where $\dot{z}$ is the generic real".

Theorem 0.33 (Woodin).

For all $x \in \mathbb{R} \cap M_1$ and Π^1_3 formulas φ, the following are equivalent:

- $\mathbb{R} \models \varphi(x)$,
- *there is an iterable $\varphi(x)$-prewitness,*
- *there is a Π^1_2-iterable $\varphi(x)$-prewitness $N \in \mathrm{HC}^{M_1}$,*
- $\mathbb{R}^{M_1} \models \psi_\varphi(x)$.

We want, for Π_2 formulas φ, a Σ_2 formula ψ_φ such that:

$$\mathcal{J}(\mathbb{R}) \models \varphi(x) \iff \mathcal{J}(\mathbb{R}^{M_{\mathrm{ld}}}) \models \psi_\varphi(x).$$

$\psi_\varphi(x)$ should say "there is a Π_1-iterable $\varphi(x)$-prewitness".

Remark 0.34.

- Π_1-iterability is $\Pi_1^{\mathcal{J}(\mathbb{R})}$.
- M_{ld} is $\Sigma_1^{\mathcal{J}(\mathbb{R})}$-correct.
- Every Π_1-iterable premouse $P \in \mathrm{HC}^{M_{\mathrm{ld}}}$ is iterable.

What is a $\varphi(x)$-prewitness (for Π_2 formulas φ)?

- Analogue to $\varphi(x)$-prewitness for Π_3^1 formulas φ?
- Not enough Woodinness in segments of M_{ld} for a direct analogue...

Definition 0.35.

An n-partial ladder is a premouse N such that for some $\vec{\theta}$,

- $\vec{\theta} = \langle \theta_i \rangle_{i \leq n}$ is a strictly increasing $(n+1)$-tuple of ordinals of N,
- θ_i is an N-cardinal for all $i \leq n$,
- θ_n^{++N} is the largest cardinal of N,
- N is closed under $M_k^\#$, for each $k < \omega$,
- $M_i^\#(N|\theta_i)$ is the Q-structure for θ_i, for each $i \leq n$, and θ_i is the least such θ.

Write $\vec{\theta}^N = \vec{\theta}$.

Definition 0.36.

Fix Σ_2 formula ϱ. Let N, θ be such that $N \models$"θ is a cardinal and θ^{++} exists" and N is $M_k^{\#}$-closed for all $k < \omega$. Write

$$S_\theta^N = T_\varrho^{N[g]}$$

for g being $(N, \mathrm{Col}(\omega, \theta))$-generic.

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for g being $(N, \mathrm{Col}(\omega, \theta))$-generic.
Given N', θ' as above with $\theta < \theta'$ and $N|\theta^{+N} = N'|\theta^{+N'}$, write

$$\pi_{\theta\theta'}^{NN'} : S_\theta^N \to S_{\theta'}^{N'}.$$

for the canonical embedding

$$\pi : T_\varrho^{N[g]} \to T_\varrho^{N'[g']},$$

where g, g' are as above with g' being $(N[g], \mathrm{Col}(\omega, \theta'))$-generic.

Fact 0.1 (Hjorth).

$S_\theta^N, S_{\theta'}^{N'}, \pi_{\theta\theta'}^{NN'}$ are independent of g, g'; so they are in N'.

Let $\varphi(x) = \neg\varrho(x)$ be Π_2. For a $\underline{\varphi(x)\text{-witness}}$, we want roughly:

– an (iterable) 0-partial ladder premouse P_0 with $x \in P_0$

where player 2 wins the following game $\mathscr{G}_x^{P_0}$:

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0.1 Player 1 plays:

- A correct tree $\mathcal{T}_0$ on P_0, based on $P_0|\theta_0^{P_0}$; let $P_0' = M_\infty^{\mathcal{T}_0}$ and $\theta_0' = \theta_0^{P_0'}$,
- $(s_0, t_0) \in (S_{\theta_0'}^{P_0'})_{\varrho(x)}$ with $\mathrm{lh}(s_0, t_0) = 1$,

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- $(s_0, t_0) \in (S_{\theta_0'}^{P_0'})_{\varrho(x)}$ with $\mathrm{lh}(s_0, t_0) = 1$,

0.2 Player 2 plays:

- A 1-partial ladder P_1 such that $P_0'|(\theta_0')^{+P_0'} \triangleleft P_1 \triangleleft P_0'$, Let $\varphi(x) = \neg\varrho(x)$ be Π_2. For a $\varphi(x)$-witness, we want roughly:

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0.2 Player 2 plays:

- A 1-partial ladder P_1 such that $P_0'|(\theta_0')^{+P_0'} \lhd P_1 \lhd P_0'$,

1.1 Player 1 plays:

- A correct tree $\mathcal{T}_1$ on P_1 which is above θ_0' and based on $P_1|\theta_1$; let $P_1' = M_\infty^{\mathcal{T}_1}$ and $\theta_1' = \theta_1^{P_1'}$,
- $(s_1, t_1) \in (S_{\theta_1'}^{P_1'})_{\varrho(x)}$ with $\pi_{\theta_0'\theta_1'}^{P_0'P_1'}(s_0, t_0) \lhd (s_1, t_1)$ and $\mathrm{lh}(s_1, t_1) = 2$,

Let $\varphi(x) = \neg\varrho(x)$ be Π_2. For a $\varphi(x)$-witness, we want roughly:

– an (iterable) 0-partial ladder premouse P_0 with $x \in P_0$

where player 2 wins the following game $\mathscr{G}_x^{P_0}$:

0.1 Player 1 plays:

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0.2 Player 2 plays:

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- $(s_1, t_1) \in (S_{\theta_1'}^{P_1'})_{\varrho(x)}$ with $\pi_{\theta_0'\theta_1'}^{P_0'P_1'}(s_0, t_0) \lhd (s_1, t_1)$ and $\mathrm{lh}(s_1, t_1) = 2$,

1.2 Player 2 plays:

- A 2-partial ladder P_2 such that $P_1'|(\theta_1')^{+P_1'} \lhd P_2 \lhd P_1'$,

Let $\varphi(x) = \neg\varrho(x)$ be Π_2. For a $\underline{\varphi(x)\text{-witness}}$, we want roughly:

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where player 2 wins the following game $\mathscr{G}_x^{P_0}$:

0.1 Player 1 plays:

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- $(s_0, t_0) \in (S_{\theta_0'}^{P_0'})_{\varrho(x)}$ with $\mathrm{lh}(s_0, t_0) = 1$,

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1.1 Player 1 plays:

- A correct tree $\mathcal{T}_1$ on P_1 which is above θ_0' and based on $P_1|\theta_1$; let $P_1' = M_\infty^{\mathcal{T}_1}$ and $\theta_1' = \theta_1^{P_1'}$,
- $(s_1, t_1) \in (S_{\theta_1'}^{P_1'})_{\varrho(x)}$ with $\pi_{\theta_0'\theta_1'}^{P_0'P_1'}(s_0, t_0) \lhd (s_1, t_1)$ and $\mathrm{lh}(s_1, t_1) = 2$,

1.2 Player 2 plays:

- A 2-partial ladder P_2 such that $P_1'|(\theta_1')^{+P_1'} \lhd P_2 \lhd P_1'$,

2.1 etc,

2.2 etc...

The first player to break a rule loses; otherwise player 2 wins.

Lemma 0.37.

If P_0 is iterable and player 2 has a winning strategy for $\mathscr{G}_x^{P_0}$ then $\mathcal{J}(\mathbb{R}) \models \varphi(x)$.

Proof.

Suppose $\mathcal{J}(\mathbb{R}) \models \neg\varphi(x)$, so $\mathcal{J}(\mathbb{R}) \models \varrho(x)$, i.e.

$$\mathcal{J}(\mathbb{R}) \models \exists^{\mathbb{R}} w\ \psi(w, x)$$

where ψ is Π_1. Let $(w, \vec{y})$ be such that $(x, w, \vec{y}) \in [T_\varrho^V]$.

Let $\vec{z} \in {}^\omega\mathbb{R}$ yield ranks of all ordinals in $\vec{y}$, w.r.t. the prewellorders of the scales.

Let $\mathcal{T}_0$ on P_0 be the $(w, \vec{z})$-genericity iteration at θ_0. Let $P'_0 = M_\infty^{\mathcal{T}_0}$. Let

$$(s_0, \widetilde{t_0}) = (w, \vec{y}) \restriction 1.$$

Let (s_0, t_0) be such that whenever g is $(P'_0, \mathrm{Col}(\omega, \theta'_0))$-generic,

$$\pi^{P'_0[g], V}(s_0, t_0) = (s_0, \widetilde{t_0}).$$

Let $P_1 \lhd P'_0$ be played by player 2.

Iterate P_1 above θ'_0, to make $(w, \vec{z})$ generic....etc.
...Tree on P_0 with unbounded drops, contradiction. □

Definition 0.38 (Pseudo-definition).

Let P_0 be a 0-partial ladder and $x \in \mathbb{R}^{P_0}$. Let $\Delta_0 \in P_0$. We say that (P_0, Δ_0) is a $\varphi(x)$-prewitness iff Δ_0 is a winning strategy in the game $\mathscr{G}_x^{*(P_0,\Delta_0)}$, which is played as is $\mathscr{G}_x^{P_0}$, except that:

- all trees $\mathcal{T}_n$ are trivial,
- Player 2 must play move $(n+1).2$ according to Δ_n,
- At move $(n+1).2$, player 2 must ensure that $\Delta_{n+1} \in P_{n+1}$, where

$$\Delta_{n+1} = \text{ tail strategy determined by } \Delta_n, s_n, t_n.$$

(See paper for formal definition.)

Definition 0.39.

Let $\langle \theta_n \rangle_{n<\omega}$ be the "rungs" of the ladder of M_{ld}. Let

$$S_\infty^{M_{\mathrm{ld}}} = \mathrm{dirlim}_{n<\omega} S_{\theta_n}^{M_{\mathrm{ld}}}$$

under the maps $\pi_{\theta_n \theta_m}^{M_{\mathrm{ld}}}$.

(Recall $\varphi(u)$ is Π_2 and $\varrho(u) \iff \neg\varphi(u)$.)

Lemma 0.40.

Let $x \in \mathbb{R}^{M_{\mathrm{ld}}}$. The following are equivalent:

- $\mathcal{J}(\mathbb{R}) \models \varphi(x)$,
- $x \notin p[T_\varrho]$,
- $(T_\varrho)_x$ *is wellfounded,*
- $x \notin p[S_\infty^{M_{\mathrm{ld}}}]$,
- $(S_\infty^{M_{\mathrm{ld}}})_x$ *is wellfounded,*
- *there is a* $\varphi(x)$*-prewitness* (P, Δ) *such that* $P \lhd M_{\mathrm{ld}}|\omega_1^{M_{\mathrm{ld}}}$,
- $M_{\mathrm{ld}} \models$*"there is a* $\varphi(x)$*-prewitness* $(P, \Delta) \in \mathrm{HC}$ *such that* P *is* Π_1*-iterable".*

(The last item gives $\psi_\varphi(x)$.)

Proof Sketch.

Suppose $x \in M_{\mathrm{ld}}$ but $x \notin p[S_\infty^{M_{\mathrm{ld}}}]$, so $(S_\infty^{M_{\mathrm{ld}}})_x$ is wellfounded.

We want a $\varphi(x)$-prewitness $P \triangleleft M_{\mathrm{ld}}|\omega_1^{M_{\mathrm{ld}}}$.

Given s, t with $\mathrm{lh}(s,t) = n$, say (P, Δ) is a $\underline{(\varphi(x), s, t)\text{-prewitness}}$ iff P is an n-partial ladder, $(s,t) \in S^P_{\theta^P_n}$, and player 2 wins from position (P, Δ, s, t).

Let

$$\pi_{\theta_n \infty} : S^{M_{\mathrm{ld}}}_{\theta_n} \to S^{M_{\mathrm{ld}}}_\infty$$

be the direct limit map.

Subclaim.

For each $n < \omega$ and each $(s,t) \in S^{M_{\mathrm{ld}}}_{\theta_n}$ with $\mathrm{lh}(s,t) = n$, there is a $(\varphi(x), s, t)$-prewitness $P \triangleleft M_{\mathrm{ld}}$ with $M_{\mathrm{ld}}|\theta_n^{+M_{\mathrm{ld}}} \triangleleft P$.

Proof.

By induction on $S^{M_{\mathrm{ld}}}_\infty$-rank of $\pi_{\theta_n\infty}(s,t)$, using condensation. □

It follows that there is a $(\varphi(x), \emptyset, \emptyset)$-prewitness $P \triangleleft M_{\mathrm{ld}}|\omega_1^{M_{\mathrm{ld}}}$. □

Theorem (S.).

Assume ZF + AD + $V = L(\mathbb{R})$. Let α be such that $[\alpha, \alpha]$ is a projective-like gap and either α is a limit of countable cofinality, or $\alpha = \beta + 1$ where β does not end a strong gap. Then:

- $\mathrm{OD}_{\alpha n} = \mathrm{OD}^{\mathbb{R}}_{\alpha n}$.
- *There is a mouse M such that $\mathrm{OD}_{\alpha n} = \mathbb{R} \cap M$.*

Proof setup.

Consider $n = 2$. The foregoing adapts to $\mathcal{J}_\alpha(\mathbb{R})$ on a certain cone of x, giving

$$\mathrm{OD}_{\alpha 2}(x) = \mathrm{OD}^{\mathbb{R}}_{\alpha 2}(x) = M^{\alpha}_{\mathrm{ld}}(x) \cap \mathbb{R}$$

for the "α-ladder" $M^{\alpha}_{\mathrm{ld}}(x)$ for such x.

For lightface version, consider (cf. [8] and [5])

$$M = \text{ output of the Q-local local } K^c\text{-construction of } M^{\alpha}_{\mathrm{ld}}(x).$$

Show

$$\mathrm{OD}_{\alpha 2} \subseteq \mathbb{R} \cap M \subseteq \mathrm{OD}^{\mathbb{R}}_{\alpha 2}.$$

Similar for $n > 2$. □

End of weak gap

Example: $[\alpha, \beta]$ is weak, and for $P_{\mathrm{g}}(x)$ the corresponding mouse on a cone of x,

$$\omega = \rho_1^{P_{\mathrm{g}}(x)} < \lambda^{P_{\mathrm{g}}(x)} < \mathrm{OR}^{P_{\mathrm{g}}(x)},$$

$\lambda^P \notin p_1^{P_{\mathrm{g}}(x)}$, $(\lambda^P)^{+P} < \mathrm{OR}^P$, and $\Sigma_1^{\mathcal{J}_\beta(\mathbb{R})}$ is μ-reflecting.(see [6]).

Definition 0.41.

For an X-premouse R, say that R is relevant if there is $\delta = \delta_0^R < \mathrm{OR}^R$ such that:

- $R \models$"δ is the least Woodin $> \mathrm{rank}(X)$",
- $R = P_{\mathrm{g}}(R|\delta)$,
- $R|\delta$ is P_{g}-closed.

Definition 0.42.

For relevant R, let:

- $\langle \alpha_n^R \rangle_{n<\omega}$ be the canonical ω-sequence cofinal in OR^R,
- $\gamma_n^R = \sup(\delta_0^R \cap \mathrm{Hull}_1^{R|\alpha_n^R}(X \cup \{p_1^R\})$,
- $t_n^R = \mathrm{Th}_1^R(X \cup \gamma_n^R \cup \{p_1^R\})$.

Definition 0.43 (Ladder mouse at end of weak gap).

For a cone of y, $M_{\mathrm{ld}}^{P_{\mathrm{g}}}(y)$ is the least relevant mouse N such that letting $\delta = \delta_0^N$, for each $n < \omega$, there is a relevant $R \triangleleft N|\delta$ with $t_n^R = t_n^N$ (after substituting p_1^R for p_1^N).

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