Order-preserving Martin's Conjecture and Inner Model Theory

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Outline

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Martin's Conjecture

There seems to be significant global structure to the functions on the Turing degrees which come up naturally in Computability Theory.

Martin's Conjecture is a precise way of stating that the global structure is really as it seems.

Roughly, the conjecture is that, under determinacy, the functions on the Turing degrees are minor variations of constant functions, the identity, and iterates of the Turing jump.

Martin's Conjecture

- For $x, y \in \mathbb{R}$, $x \leq_T y$ iff x is computable from y and $x \equiv_T y$ iff x is computable from y and vice-versa.
- A *cone* is a set of the form $\{x \in \mathbb{R} \mid x \geq_T b\}$ for some real b.
- A set of reals is *Turing-invariant* iff it is closed under $\equiv_T$.
- A function $f : \mathbb{R} \to \mathbb{R}$ is *Turing-invariant* iff $x \equiv_T y \Rightarrow f(x) \equiv_T f(y)$.

A Turing-invariant function induces a function on the Turing degrees. Every function on the Turing degrees arises in this way under AC but also under AD^+.

Theorem (Martin, '68)

Assume AD. *Any Turing-invariant set of reals either contains a cone or is disjoint from a cone.*

Martin's Conjecture

Definition

For $f, g : \mathbb{R} \to \mathbb{R}$, put $f \leq_M g$ iff $f(x) \leq_T g(x)$ on a cone of $x \in \mathbb{R}$ and $f \equiv_M g$ iff $f(x) \equiv_T g(x)$ on a cone of $x \in \mathbb{R}$.

We can now state Martin's Conjecture.

Assume $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$.

1. For any Turing-invariant function $f : \mathbb{R} \to \mathbb{R}$, either there is a $c \in \mathbb{R}$ such that $f(x) \equiv_T c$ on a cone or $f(x) \geq_T x$ on a cone.
2. $\leq_M$ prewellorders the Turing-invariant functions f such that $f(x) \geq_T x$ on a cone and the successors in this prewellorder are given by composition with the Turing jump.

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Order-preserving Martin's Conjecture

Both parts of Martin's Conjecture are still open, even for Borel functions, but there have been partial results.

Slaman and Steel showed both parts of Martin's Conjecture hold for the class of *uniformly* Turing-invariant functions, which is strong evidence for the truth of the conjecture.

In this talk, we'll consider Martin's Conjecture restricted to the class of *order-preserving* functions, which turns out to be more tractable.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is *order-preserving* iff $x \leq_T y \Rightarrow f(x) \leq_T f(y)$.

Order-preserving Martin's Conjecture

Slaman and Steel also showed

Theorem (Slaman-Steel, '88)

Assume ZF*. Part 2 of Martin's Conjecture holds for Borel order-preserving functions.*

More recently, we showed that part 1 of Martin's Conjecture is true for order-preserving functions.

Theorem (Lutz-S., 2021)

Assume AD*. For any order-preserving $f : \mathbb{R} \to \mathbb{R}$, either*

- *there is a c such that $f(x) \equiv_T c$ on a cone, or*
- *$f(x) \geq_T x$ on a cone.*

Combining this with the Slaman-Steel result, both parts of Martin's Conjecture hold for Borel order-preserving functions. Moreover, this is actually provable from just ZF.

Order-preserving part 2

In the rest of the talk we'll discuss some recent progress on part 2 of Martin's Conjecture for order-preserving functions and an as yet unrealized plan for proving part 2 of Martin's Conjecture under Mouse Capturing (MC).

Recall that Slaman-Steel showed that part 2 of Martin's Conjecture holds for Borel order-preserving functions. Combining their work with a result of Woodin gives a bit more.

Theorem (Slaman-Steel-Woodin, '88)

Assume AD*. Let $f : \mathbb{R} \to \mathbb{R}$ be order-preserving such that $f(x) \geq_T x$ on a cone. Then either $f(x) \equiv_T x^{(\alpha)}$ on a cone for some $\alpha < \omega_1$ or $f(x) \geq_T \mathcal{O}^x$ on a cone.*

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Main theorem

Recently, we showed

Theorem (Lutz-S.)

Assume AD. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be order-preserving such that* $f(x) \geq_T \mathcal{O}^x$ *on a cone. Then either* $f(x) \equiv_T (\mathcal{O}^x)^{(\sigma(\omega_1^x))}$ *on a cone for some* $\sigma : \omega_1 \to \omega_1$ *or* $f(x) \geq_T \mathcal{O}^{\mathcal{O}^x}$ *on a cone.*

Definition

- A function $g : \mathbb{R} \to \omega_1$ is *Turing-invariant* iff $x \equiv_T y \Rightarrow g(x) = g(y)$.
- For $g, h : \mathbb{R} \to \omega_1$, put $g \leq_M h$ iff $g(x) \leq h(x)$ on a cone.

Martin's Cone Theorem implies that Turing-invariant functions $g, h : \mathbb{R} \to \omega_1$ are prewellordered by $\leq_M$.
Since functions of the form $x \mapsto \sigma(\omega_1^x)$ are Turing-invariant, our theorem implies part 2 of Martin's Conjecture holds for order-preserving functions below $x \mapsto \mathcal{O}^{\mathcal{O}^x}$.

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Mouse operators

We'll take an inner-model-theoretic perspective on our theorem which we believe is important for trying to generalize it.

Jensen used his fine-structure to identify canonical constructible reals, the mastercodes for levels of L. Rudominer extended these results to ms-indexed premice.

Theorem (Rudominer, '98)

Let $\mathcal{M}$ be a sound x-premouse and suppose there is a $\Delta_{n+1}(\mathcal{M})$ real which is not $\Delta_n(\mathcal{M})$. Then there is one of maximum Turing degree, i.e. there is an $y \in \mathbb{R}$ which is $\Delta_{n+1}(\mathcal{M})$ such that every x that is $\Delta_{n+1}(\mathcal{M})$ is computable from y.

We call such a real y a $\Delta_{n+1}(\mathcal{M})$ *mastercode*. For our previous theorem, we only need to consider x-premice which are levels of L, i.e. of the form $J_\alpha[x]$ for some x.

Mouse operators

Definition

- A *mouse operator* is a function M with domain $\mathbb{R}$ such that for all $x \in \mathbb{R}$,
  - $M(x)$ is a sound ω_1-iterable x-premouse
  - for any $y \equiv_T x$, $M(y)$ is the reorganization of $M(x)$ as a y-premouse.
- A mouse operator M is *relevant* iff for all $x \in \mathbb{R}$ there is some $n \in \omega$ such that there is a $\Delta_{n+1}(M(x))$ real which is not $\Delta_n(M(x))$.
- Given a relevant mouse operator M we let f_M denote any function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is a $\Delta_{n+1}(M(x))$ mastercode for n least such that there is a $\Delta_{n+1}(M(x))$ real which is not $\Delta_n(M(x))$.

Given a relevant mouse operator M, f_M is unique up to $\equiv_M$. One can actually take f_M to be uniformly Turing-invariant. Also note that the relevant mouse operators are naturally prewellordered by the relation $N(x) \trianglelefteq M(x)$ on a cone, since relevant mouse operators output sound $\omega_1 + 1$-iterable x-premice which project to ω.

Mouse operators

Here are some examples of relevant mouse operators M and associated functions f_M.

- $x \mapsto J_1[x]$ and $x \mapsto x^{(\omega)}$,
- $x \mapsto J_{\omega_1^x}[x]$ and $x \mapsto \mathcal{O}^x$,
- $x \mapsto J_{\omega_1^x+1}[x]$ and $x \mapsto (\mathcal{O}^x)^{(\omega)}$,
- $x \mapsto J_{\sigma(\omega_1^x)}[x]$ and $x \mapsto (\mathcal{O}^x)^{(\omega\cdot(\sigma(\omega_1^x)-\omega_1^x))}$, for nondecreasing $\sigma : \omega_1 \to \omega_1$,
- $x \mapsto J_{\omega_2^x}[x]$ and $x \mapsto \mathcal{O}^{\mathcal{O}^x}$,
- $x \mapsto$ the premouse $x^\#$ and $x \mapsto$ the real $x^\#$,
- $x \mapsto M_1^\#(x)$, ...

Notice that the way we've set things up we're only getting the functions of limit rank among the natural Turing-invariant functions. One gets the successors by composing with the Turing jump.

Mouse operators

From Slaman and Steel's work, we can identify sufficient criteria for establishing part 2 of Martin's Conjecture for order-preserving functions below some f_M for M a relevant mouse operator.

These criteria are just generalizations of well-known theorems from computability theory about the Turing jump to the mouse operators up to M.

Theorem (Friedberg, '57)

Let $x, y \in \mathbb{R}$ and suppose $y \geq_T x'$. Then there is a $z \geq_T x$ such that $y \equiv_T z'$.

Definition

A relevant mouse operator M has the *jump inversion property* iff on a cone of x, for all $y \geq_T f_M(x)$ there is a $z \geq_T x$ such that $y \equiv_T f_M(z)$.

Mouse operators

Moreover, Friedberg's proof for the Turing jump provides a general method of establishing the jump inversion property

Fact (Friedberg, essentially)

Suppose that on a cone of x $M(x \oplus y) = M(x)[y]$ whenever y is a sufficiently Cohen generic real. Then M has the jump inversion property.

Consider $M(x) = J_{\sigma(\omega_1^x)}[x]$ for some non-decreasing $\sigma : \omega_1 \to \omega_1$.

If y is sufficiently Cohen generic over $J_{\sigma(\omega_1^x)}[x]$, then $\omega_1^{x \oplus y} = \omega_1^x$ because sufficiently generic Cohen reals preserve admissibility! It follows that $\sigma(\omega_1^{x \oplus y}) = \sigma(\omega_1^x)$ and so $J_{\sigma(\omega_1^{x \oplus y})}[x \oplus y] = J_{\sigma(\omega_1^x)}[x][y]$.

So M has the jump inversion property.

Mouse operators

This property is useful because it can take us through successor steps of an inductive argument verifying instances of part 2 of Martin's Conjecture for order-preserving functions.

Lemma

Let M be a relevant mouse operator with the jump inversion property, $n \in \omega$, and g be an order-preserving function such that $g(x) >_T f_M(x)^{(n)}$ on a cone. Then $g(x) \geq_T f_M(x)^{(n+1)}$ on a cone.

The next property will get us through limit steps and comes from abstracting the Posner-Robinson theorem.

Theorem (Posner-Robinson, '81)

Let $x, y \in \mathbb{R}$, and suppose $y \not\leq_T x$. Then there is $z \geq_T x$ such that $y \oplus z \geq_T z'$.

Mouse operators

Definition

A relevant mouse operator M has the *Posner-Robinson property* iff on a cone of x, for all $y \geq_T x$ such that $y \notin M(x)$, there is a $z \geq_T x$ such that $y \oplus z \geq_T f_M(z)$.

The original proof of Posner-Robinson does not generalize but Kumabe and Slaman found a forcing proof which does generalize.

Fact (Kumabe-Slaman, essentially)

Suppose that on a cone of x, $M(x \oplus y) = M(x)[y]$ whenever y is a sufficiently Kumabe-Slaman generic real. Then M has the Posner-Robinson property.

Mouse operators

Consider $M(x) = J_{\sigma(\omega_1^x)}[x]$ for some non-decreasing $\sigma : \omega_1 \to \omega_1$.

If y is sufficiently Kumabe-Slaman generic, then $\omega_1^{x \oplus y} = \omega_1^x$ because sufficiently generic Kumabe-Slaman reals preserve admissibility, too. As before, we get $J_{\sigma(\omega_1^{x \oplus y})}[x \oplus y] = J_{\sigma(\omega_1^x)}[x][y]$, so M has the Posner-Robinson property.

Preservation of admissibility similarly implies that $M(x) = J_{\omega_2^x}[x]$ has the Posner-Robinson property.

Lemma (Slaman-Steel)

Suppose M has the Posner-Robinson property. Then for every order-preserving g either $g(x) \in M(x)$ on a cone or $g(x) \geq_T f_M(x)$ on a cone.

Mouse operators

These results identify a way one might try to prove part 2 of Martin's Conjecture from Mouse Capturing.

Theorem

Assume $\mathrm{AD}^+ + \mathrm{MC}$*. Suppose every relevant mouse operator M has the jump inversion property and the Posner-Robinson property. Then part 2 of Martin's Conjecture holds for all order-preserving functions.*

Proof sketch. We show that every order-preserving function is actually uniformly Turing-invariant, so that part 2 of Martin's Conjecture holds for the order-preserving functions by Steel's theorem.

If this failed, then there would be an OD_x order-preserving g which is not uniformly Turing-invariant, by AD^+. By MC, $g(x) \in M(x)$ on a cone for some relevant mouse operator M.

Mouse operators

Let M be the least such operator. Then M must actually have successor rank, i.e. be of the from $J_1(N(x))$ for some N. To see this, let $N(x)$ be the least proper initial segment of $M(x)$ such that $g(x)$ is definable over $N(x)$. Then N is a relevant mouse operator and $g(x) \in J_1(N(x))$. So $J_1(N(x)) = M(x)$, by our minimality hypothesis on M. It follows that $\{f_N(x)^{(n)} \mid n \in \omega\}$ is $\leq_T$-cofinal in the reals of $M(x)$.

Since $g(x) \notin N(x)$ on a cone, $g(x) \geq_T f_N(x)$ on a cone, since N has the Posner-Robinson property. Since $g(x) \in M(x)$ on a cone, we cannot have that $g(x) >_T f_N(x)^{(n)}$ for all n. So let n be least such that $g(x) \not>_T f_N(x)^{(n)}$. Then the jump inversion property for N gives that $g(x) \equiv_T f_N(x)^{(n)}$. So g is uniformly Turing-invariant after all, a contradiction. □

Outline

Proof of main theorem

We'll end the talk by giving a rough outline of how to prove our main theorem.

Theorem (Lutz-S.)

Assume AD*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be order-preserving such that* $f(x) \geq_T \mathcal{O}^x$ *on a cone. Then either* $f(x) \equiv_T (\mathcal{O}^x)^{(\sigma(\omega_1^x))}$ *on a cone for some* $\sigma : \omega_1 \to \omega_1$ *or* $f(x) \geq_T \mathcal{O}^{\mathcal{O}^x}$ *on a cone.*

Let $M(x) = J_{\omega_2^x}[x]$. We saw that this M has the Posner-Robinson property and that $f_M(x) = \mathcal{O}^{\mathcal{O}^x}$. So it suffices to show that if $f(x) \in J_{\omega_2^x}[x]$, then $f(x) \equiv_T (\mathcal{O}^x)^{(\sigma(\omega_1^x))}$ on a cone for some non-decreasing σ.

For this, it is enough to show that the relevant mouse operations N such that $J_{\omega_1^x}[x] \trianglelefteq N(x) \triangleleft J_{\omega_2^x}[x]$ are exactly the $J_{\sigma(\omega_1^x)}[x]$ for some nondecreasing $\sigma : \omega_1 \to \omega_1$.

Proof of main theorem

To see that this suffices, we just run through the argument just sketched, using that for $N(x) = J_{\sigma(\omega_1^x)}[x]$ we know that N has the jump inversion property and the Posner-Robinson property and that $f_N(x) = (\mathcal{O}^x)^{(\omega\cdot(\sigma(\omega_1^x)-\omega_1^x))}$.

Okay, suppose N such that $J_{\omega_1^x}[x] \trianglelefteq N(x) \triangleleft J_{\omega_2^x}[x]$. Then we have $N(x) = J_{g(x)}[x]$ for some Turing-invariant $g : \mathbb{R} \to \omega_1$ such that $\omega_1^x \leq g(x) < \omega_2^x$ on a cone. So we just need to establish the following lemma.

Lemma

Assume AD. *Let* $g : \mathbb{R} \to \omega_1$ *be such that* $\omega_1^x \leq g(x) < \omega_2^x$. *Then* $g(x) = \sigma(\omega_1^x)$ *for some non-decreasing* $\sigma : \omega_1 \to \omega_1$.

Proof of main theorem

Proof sketch.
The first two generators of the short extender of the Martin measure ultrapower are ω_1 and ω_2, since it must be the extender of an iterated ultrapower by the club filter on ω_1.

Martin showed $x \mapsto \omega_1$ represents ω_1 in this ultrapower. Steel showed $x \mapsto \omega_2^x$ represents ω_2. So for g such that $\omega_1^x \leq g(x) < \omega_2^x$, g represents an ordinal α which is *not* a generator.

Since ω_1 is the only generator below g and $x \mapsto \omega_1^x$ represents ω_1, there must be a $\sigma : \omega_1 \to \omega_1$ such that $g(x) = \sigma(\omega_1^x)$. □

The End

Thanks!