Tutorial on the Mouse Set Conjecture

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We outline a proof of Strong Mouse Capturing in natural models of AD^+ (i.e. those of the form $V = L(\wp(\mathbb{R})))$ below the minimal model of LSA, which is the theory $\mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$ and θ_α is the largest Suslin cardinal. Basic terminology and definitions concerning hod mice are taken from [5, 7].

Definition 0.1 Strong Mouse Capturing *(*SMC*) is the statement that for any hod pair or an sts hod pair* $(\mathcal{P}, \Sigma)$ *such that* Σ *has strong branch condensation and is strongly fullness preserving, and for any reals* x, y*,* x *is ordinal definable from* Σ *and* y *if and only if* x *is in some* Σ*-mouse over* y*.*

Definition 0.2 $\#_{lsa}$ *is the statement: there is a pointclass* $\Gamma \subseteq \wp(\mathbb{R}))$ *such that* $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$ *and there is a Suslin cardinal bigger than* $w(\Gamma)$*.*

We show

Theorem 0.3 *Assume* $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}) + \neg\#_{lsa}$*. Then the* Strong Mouse Capturing *holds.*

1 Outline of the proof

Towards a contradiction assume that SMC is false. Our first step is to locate the minimal level of the Wadge hierarchy over which SMC becomes false. For simplicity we assume that the Mouse Capturing, instead of the Strong Mouse Capturing, is false. Mouse Capturing is the same as SMC when the pair $(\mathcal{P}, \Sigma) = \emptyset$. The general case is only different in one aspect, it needs to be relativized to some strategy or a short tree strategy Σ.

Let Γ be the least Wadge initial segment such that for some α

1. $\Gamma = \wp(\mathbb{R}) \cap L_\alpha(\Gamma, \mathbb{R})$,
2. $L_\alpha(\Gamma, \mathbb{R}) \vDash \mathsf{SMC}$,
3. there are reals x and y such that $L_{\alpha+1}(\Gamma, \mathbb{R}) \vDash$ "y is $OD(x)$" yet no x-mouse has y as a member.

See [5] for definitions of $\mathbb{B}(Q^-, \Sigma_{Q^-})$, $B(\mathcal{P}, \Sigma)$, strongly guided etc.

Definition 1.1 *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair and* Γ^* *is a projectively closed pointclass. We say* $(\mathcal{P}, \Sigma)$ *is* Γ^****-perfect*** *if the following conditions are met.*

1. Σ *is* Γ^**-strongly fullness preserving and has strong branch condensation.*
2. *For every* $Q \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$ *such that* Q *is of successor type, there is* $\vec{B} = (B_i : i \leq \omega) \subseteq \mathbb{B}(Q^-, \Sigma_{Q^-})$ *such that* $\vec{B}$ *strongly guides* Σ_Q*.*

If $\Gamma^* = \wp(\mathbb{R})$ *then we omit* Γ^* *from our notation.* ⊣

The following theorem was heavily used in [4]. It is essentially due to Steel and Woodin (see [2]).

Theorem 1.2 *Assume* AD^+ *and suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair or an sts hod pair*[1] *(or an anomalous pair) such that* $L(\Sigma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma)$ *is perfect". Then* $L(\Sigma, \mathbb{R}) \vDash \mathsf{MC}(\Sigma)$*. Furthermore, for every* $\mathcal{R} \lhd^c_{hod} \mathcal{P}$, $L(\Sigma, \mathbb{R}) \vDash \mathsf{MC}(\Sigma_{\mathcal{R}})$.

A key theorem used in the proof of Theorem 0.3 is the following capturing theorem. Its precursor is stated as [4, Theorem 6.5].

Theorem 1.3 *Suppose* $(\mathcal{P}, \Sigma)$ *is a perfect hod pair and* Γ_1 *is a good pointclass such that* $\mathsf{Code}(\Sigma) \in \Delta_{\Gamma_1}$*. Suppose* F *is as in Theorem 2.7 for* Γ_1 *and* $z \in \mathrm{dom}(F)$ *is such that if* $F(z) = (\mathcal{N}^*_z, \mathcal{M}_z, \delta_z, \Sigma_z)$ *then* $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ *Suslin, co-Suslin captures* $\mathsf{Code}(\Sigma)$[2]*. Let* $\mathcal{N} = (\mathsf{Le}(\emptyset))^{\mathcal{N}^*_z | z}$[3]*. Then there is* $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ *such that* $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in L[\mathcal{N}]$.

The next key lemma that is used in the proof of Theorem 0.3 is the following generation lemma that can be traced to [5, Lemma 6.23]. Below Γ is as above.

Lemma 1.4 *There is a perfect pair* $(\mathcal{P}, \Sigma)$ *such that*

$$\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R}).$$

Suppose now that $(\mathcal{P}, \Sigma)$ is a Γ-perfect pair such that $\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R})$. Such a pair is given to us by Lemma 1.4.

We now apply Theorem 1.2. For each $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ there is a $\Sigma_{\mathcal{Q}}$-mouse $\mathcal{M}_{\mathcal{Q}}$ over $(\mathcal{Q}, x)$ such that y is definable over $\mathcal{M}_{\mathcal{Q}}$. We then again can find an x-mouse $\mathcal{N}$ such that for some $\mathcal{Q} \in \mathcal{N} \cap pI(\mathcal{P}, \Sigma)$, $\mathcal{M}_{\mathcal{Q}} \in \mathcal{N}$. It follows that $y \in \mathcal{N}$. Thus, to finish the proof of Theorem 0.3, it is enough to establish Theorem 1.3 and Lemma 1.4.

2 Γ-Woodin mice

We recall the definition of a good pointclass (see [14, Definition 9.12]). Unlike [14, Definition 9.12] we include scale property into the definition of good pointclass.

Definition 2.1 *We say* Γ *is a good pointclass if* Γ *is closed under recursive substitutions, is closed under quantification over* ω*, is closed under existential quantification over* $\mathbb{R}$*, is* ω*-parametrized*[4] *and has the scale property.* ⊣

Suppose Γ is a good pointclass. For $x \in \mathbb{R}$, we let $C_\Gamma(x)$ be the largest countable $\Gamma(x)$-set of reals. For transitive $a \in \mathsf{HC}$[5] and surjection $g : \omega \to a$, we let a_g be the real coding $(a, \in)$ via g. More precisely,

$$a_g(k) = \{ 1 : k = 2^m 3^n \text{ and } g(m) \in g(n)$$

0 : otherwise. Clearly $M_{a_g} = (a, \in)$. If $b \subseteq a$, then we let $b_g = \{m : g(m) \in b\}$. We then let $C_\Gamma(a) = \{b \subseteq a : \text{for comeager many } g : \omega \to a, b_g \in C_\Gamma(a_g)\}$.

Continuing with Γ, we say P is a Γ-Woodin if there is a P-cardinal δ_P such that

[1] In the case $(\mathcal{P}, \Sigma)$ is an sts pair and there is no Suslin cardinal above Σ (like in the minimal model of LSA and Σ has Wadge rank the largest Suslin cardinal), we only can prove sommething like the "Furthermore" clause. More precisely, the proof shows that for any $(\mathcal{Q}, \Lambda) \in I^b(\mathcal{P}, \Sigma)$ (see [7] for this notation), $L(\Lambda_{\mathcal{Q}^b}, \mathbb{R}) \vDash$ "for any $R \trianglelefteq^c_{hod} \mathcal{Q}^b$, $\mathsf{MC}(\Lambda_{\mathcal{R}})$".TheproofalsoshowsthatL$(\Sigma, \mathbb{R}) \vDash \mathsf{MC}(\Sigma)$ if there is a Suslin cardinal above Σ in the model.

[2] We abuse the terminology and omit the other object used to express this type of capturing. In the sequel, if the nature of these other objects, like the pair (N, Ψ), is not important we will omit them from the discussions.

[3] This is just the ordinary fully backgrounded construction.

[4] This means that there is $U \subseteq \omega \times \mathbb{R}$ such that $U \in \Gamma$ and $\{A \subseteq \mathbb{R} : A \in \Gamma\} = \{U_e : e \in \omega\}$.

[5] HC is the set of hereditarily countable sets.

1. P is countable,

2. $P = C_\Gamma(C_\Gamma(V^P_{\underset{\cdot}{P}}))$,

3. $P \models$ "δ_P is the only Woodin cardinal" and

4. for every $\eta < \underset{\cdot}{P}$, $C_\Gamma(V_\eta) \models$ "η is not a Woodin cardinal".

We say (P, Ψ) is a Γ-*Woodin pair* if

1. Ψ is an ω_1-iteration strategy for P and

2. for every Ψ-iterate Q of P, Q is a Γ-Woodin[6].

Woodin, assuming AD^+, showed that if Γ is a good pointclass not closed under $\forall^{\mathbb{R}}$ then there are Γ-Woodin pairs (see [14, Theorem 10.3]).

Suppose Γ is a good pointclass and (P, Ψ) is a Γ-Woodin pair. Let $\mathcal{L}_\Psi$ be the extension of the language of set theory obtained by adding one predicate symbol $\dot\Psi$ and one constant symbol e. The intended interpretation of $\dot\Psi$ is $\mathsf{Code}(\Psi)$. e wlll denote a real number. Given $u \in \mathbb{R}$, we define $T'_n(\Psi, u)$ to be the set of $(\phi, \vec{x})$ such that ϕ is a Σ_n-formula in $\mathcal{L}_\Psi$, $\vec{x} \in \mathbb{R}^m$ where m is the number of free variables of ϕ and

$$(\mathsf{HC}, \mathsf{Code}(\Psi), u, \in) \models \phi[\vec{x}].$$

We let $T'_n(\Psi) = T'_n(\Psi, 0)$.

Next we code $T'_n(\Psi, u)$ by a set of reals as follows. First let G_Ψ be the set of natural numbers that are Gödel numbers for $\mathcal{L}_\Psi$-formulae. We say $y \in \mathbb{R}$ is Ψ-appropriate if $y(0)$ is a Gödel number of an $\mathcal{L}_\Psi$ formula. If y is Ψ-appropriate then we let ϕ_y be the formula that $y(0)$ codes and l_y be the number of free variables of ϕ_y. Let $(p_i : i < \omega)$ be the sequence of prime numbers in increasing order. For $i \leq l_y$, let $y_i \in \mathbb{R}$ be such that for all $k \in \omega$, $y_i(k) = y(p_i^{k+1})$. If y is Ψ-appropriate then we say y is neat if for all k' such that $k' \neq 0$ and $k' \notin \{p_i^k : i < l_y \wedge k \in \omega\}$, $y(k') = 0$. Let then $T_n(\Psi, u)$ be the set of Ψ-appropriate neat $y \in \mathbb{R}$ such that

$$(\phi_y, \mathsf{merge}(y_i : i < l_y)) \in T'_n(\Psi, u).$$

Again, set $T_n(\Psi) = T_n(\Psi, 0)$.

Suppose $z \in \mathbb{R}$, ϕ is an $\mathcal{L}_\Psi$-formula with $l + 1$ free variables and $(x_i : 2 \leq i \leq l) \in \mathbb{R}^m$. Let $y_0 \in \mathbb{R}$ be such that $y_0(0)$ is the Gödel number of ϕ and for $i > 0$, $y_0(i) = 0$. Let $y_1 = z$ and for $2 \leq i \leq l$, $y_i = x_i$. Set $a(\phi, z, \vec{x}) = \mathsf{merge}((y_i : i \leq l))$. Notice that $(\phi, z, \vec{x})$ is uniquely determined by $a(\phi, z, \vec{x})$. In fact, the function $(\phi, z, \vec{x}) \mapsto a(\phi, z, \vec{x})$ is a Π^0_1 injection.

Assuming AD, if $A \subseteq \mathbb{R}$ then $w(A)$ is its Wadge rank, and if Γ is a pointclass then $w(\Gamma) = \sup\{w(A) : A \in \Gamma\}$.

Notation 2.2 *Suppose Γ is a pointclass closed under continous preimages and $A \subseteq \mathbb{R}$. We say A is a least upper bound for Γ if $\Gamma = \{B \subseteq \mathbb{R} : w(B) < w(A)\}$. Set then $lub(\Gamma) = \{A \subseteq \mathbb{R} : A$ is a least upper bound for $\Gamma\}$.* ⊣

Definition 2.3 *Suppose Γ is any pointclass closed under the continuous preimages. We say that the tuple $(\mathbb{M}, (\mathcal{P}, \Psi), \Gamma^*, A)$ Suslin, co-Suslin captures Γ if the following conditions hold:*

1. $A \in lub(\Gamma)$,

[6] P is a coarse structure, there is no notion of dropping for iterations of P, so P-to-Q embedding always exists.

2. Γ^* *is the least good pointclass such that* $\Gamma \subseteq \Delta_{\Gamma^*}$.

3. (P, Ψ) *is a* Γ^**-Woodin pair.*

4. (P, δ_P, Ψ) *Suslin, co-Suslin captures* A.

5. $\mathbb{M}$ *is a self-capturing background as defined in [7, Definition 4.1.5].*

6. $\mathbb{M}$ *Suslin, co-Suslin captures the sequence* $(T_n(\Psi) : n < \omega)$.

$\dashv$

Notation 2.4 *Suppose* Γ *is a pointclass closed under the continuous preimages,* $\mathsf{C} = (\mathbb{M}, (\mathcal{P}, \Psi), \Gamma^*, A)$ *Suslin, co-Suslin captures* Γ *and* $\mathbb{M} = (M, \underset{\cdot}{,} \vec{G}, \Sigma)$. *If* N *is a* Σ*-iterate of* M *then we set* $\mathsf{C}_N = (\mathbb{M}_N, (\mathcal{P}, \Psi), \Gamma^*, A)$. $\dashv$

Terminology 2.5 *We say that "g is* $< \eta$*-generic" to mean that the poset for which g is generic has size* $< \eta$. *Similarly we say that "g is* $\leq \eta$ *generic" to mean that the poset for which g is generic has size* $\leq \eta$. $\dashv$

Lemma 2.6 (Correctness of backgrounds) *Suppose* $(\mathbb{M}, (\mathcal{P}, \Psi), \Gamma^*, A)$ *Suslin, co-Suslin captures* Γ *and set* $\mathbb{M} = (M, \underset{\cdot}{,} \vec{G}, \Sigma)$. *Suppose* $x \in \mathbb{R} \cap M$. *Let* $(S_n, U_n : n < \omega) \in M$ *be the sequence of trees on* $\omega \times (\delta^+)^M$ *such that* (S_n, U_n) *Suslin, co-Suslin captures* $T_n(\Psi)$. *Let* g *be* $< \delta$*-generic over* M. *Then for any real* $u \in M[g]$,

$$(\mathsf{HC}^{M[g]}, \mathsf{Code}(\Psi) \cap M[g], u, \in) \prec (\mathsf{HC}, \mathsf{Code}(\Psi), u, \in).$$

Self-capturing backgrounds are very useful for building hod pairs and proving comparison. The following theorem of Woodin shows that under AD^+, self-capturing backgrounds are abundant.

Theorem 2.7 (Woodin, Theorem 10.3 of [14]) *Assume* AD^+. *Suppose* Γ *is a good pointclass and there is a good pointclass* Γ^* *such that* $\Gamma \subseteq \Delta_{\Gamma^*}$. *Suppose* (N, Ψ) *is* Γ^**-Woodin which Suslin, co-Suslin captures some* $A \in lub(\Gamma)$. *There is then a function* F *defined on* $\mathbb{R}$ *such that for a Turing cone of* x, $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$ *is such that*

1. $N \in L_1[x]$,

2. $\mathcal{N}^*_x | \delta_x = \mathcal{M}_x | \delta_x$,

3. $\mathcal{M}_x$ *is a* Ψ*-mouse over* x*: in fact,* $\mathcal{M}_x = \mathcal{M}_1^{\Psi,\#}(x) | \kappa_x$ *where* κ_x *is the least inaccessible cardinal of* $\mathcal{M}_1^{\Psi,\#}(x)$ *that is* $> \delta_x$,

4. $\mathcal{N}^*_x \vDash$ *"*δ_x *is the only Woodin cardinal",*

5. Σ_x *is the unique iteration strategy of* $\mathcal{M}_x$,

6. $\mathcal{N}^*_x = L(\mathcal{M}_x, \Lambda)$ *where* $\Lambda = \Sigma_x \restriction \mathrm{dom}(\Lambda)$ *and*

$$\mathrm{dom}(\Lambda) = \{\mathcal{T} \in \mathcal{M}_x : \mathcal{T} \textit{ is a normal iteration tree on } \mathcal{M}_x, \mathrm{lh}(\mathcal{T}) \textit{ is a limit ordinal and } \mathcal{T} \textit{ is below } \delta_x\},$$

7. *setting* $\vec{G} = \{(\alpha, \vec{E}^{\mathcal{N}^*_x}(\alpha)) : \mathcal{N}^*_x \vDash \text{"}\mathrm{lh}(\vec{E}^{\mathcal{N}^*_x}(\alpha)) \textit{ is an inaccessible cardinal } < \delta_x\text{"}\}$ *and* $\mathbb{M}_x = (\mathcal{N}^*_x, \delta_x, \vec{G}, \Sigma_x)$, $(\mathbb{M}_x, (N, \Psi), \Gamma^*, A)$ *Suslin, co-Suslin captures* Γ[7].

[7] Hence, $(\mathcal{N}^*_x, \delta_x, \vec{G}, \Sigma_x)$ is a self-capturing background.

3 Hod mice

For hod mice below $\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular, the definition of hod mice is given in [5]. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma_\alpha^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

1. $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_\alpha^{\mathcal{P}}$ is Woodin;
2. every Woodin cardinal or limit of Woodin cardinals of $\mathcal{P}$ is of the form $\delta_\alpha^{\mathcal{P}}$ for some α;
3. $\mathcal{P}(0) = Lp_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp_\omega^{\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_{\alpha+1}))^{\mathcal{P}}$;[8] for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_\omega^{\oplus_{\beta<\alpha}\Sigma_\beta^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
4. $\mathcal{P} \vDash \Sigma_\alpha^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$[9]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;
5. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_\beta^{\mathcal{P}}$ extends $\Sigma_\alpha^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_\beta^{\mathcal{P}}$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a *hod initial segment* of $\mathcal{Q}$. We say $(\mathcal{P}, \Sigma)$ is a *hod pair* if $\mathcal{P}$ is a hod premouse and Σ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, (strong) branch condensation, and is (strongly) Γ-fullness preserving for some pointclass Γ.

We say that M is a minimal model of LSA if

1. $M \vDash \mathsf{LSA}$,
2. $M = L(A, \mathbb{R})$ for some $A \subseteq \mathbb{R}$, and
3. for any $B \in \wp(\mathbb{R}) \cap M$ such that $w(B) < w(A)$, $L(B, \mathbb{R}) \vDash \neg\mathsf{LSA}$.

It makes sense to talk about "the" minimal model of LSA. When we say M is *the minimal* model of LSA we mean that M is a minimal model of LSA and $Ord, \mathbb{R} \subseteq M$. Clearly from the prospective of a minimal model of LSA, the universe is the minimal model of LSA. The proof of [7, Theorem 10.3.1] implies that there is a unique minimal model of LSA such that $Ord, \mathbb{R} \subseteq M$[10]. This unique minimal model of LSA is *the* minimal model of LSA.

One of the main contributions of [7] is the detailed description of V_Θ^{HOD} assuming that the universe is the minimal model of LSA. The early chapters of [7] deal with what is commonly referred to as the *HOD* analysis. These early chapters introduce the notion of a *short-tree-strategy* mouse, which is the most important technical notion studied by [7]. To motivate the need for this concept, we first recall some of the other aspects of the *analysis*.

Recall the Solovay Sequence (for example, see [5, Definition 0.9] or [16, Definition 9.23]). Recall that Θ is the least ordinal that is not a surjective image of the reals. The Solovay Sequence is a way of measuring the complexity of the surjections that can be used to map the reals onto the ordinals below Θ. Assuming AD, let $(\theta_\alpha : \alpha \leq \Omega)$ be a closed in Θ sequence of ordinals such that

1. θ_0 is the least ordinal η such that $\mathbb{R}$ cannot be mapped surjectively onto η via an ordinal definable function,

[8] $\mathcal{P}(\alpha+1)$ is a (g-organized) Σ_α-premouse in the sense defined above.

[9] This just means $\Sigma_\alpha^{\mathcal{P}}$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

[10] This proof of [7, Theorem 10.3.1] shows that the common part of a divergent models of *AD* contains a minimal model of LSA.

2. for $\alpha + 1 \leq \Omega$, fixing a set of reals A such that A has Wadge rank θ_α, $\theta_{\alpha+1}$ is the least ordinal η such that $\mathbb{R}$ cannot be mapped surjectively onto η via a function that is ordinal definable from A,

3. for limit ordinal $\lambda \leq \Omega$, $\theta_{\text{l}=\sup_{\alpha<\text{l}}\theta_\alpha}$, and

4. Ω is least such that $\theta_\Omega = \Theta$.

It follows from the definition of LSA that if κ is the largest Suslin cardinal then it is a member of the Solovay Sequence. It is not hard to show that LSA is a much stronger axiom than $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular". Under LSA, letting κ be the largest Suslin cardinal, there is an ω-club $C \subseteq \kappa$ such that for every $\lambda \in C$, $L(\Gamma_\lambda, \mathbb{R}) \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \lambda = \Theta + \Theta$ is regular", where $\Gamma_\lambda = \{A \subseteq \mathbb{R} : w(A) < \lambda\}$.[11]

Assume now that V is the minimal model of LSA. It follows from the work done in [7] that for every κ that is a member of the Solovay Sequence but is not the largest Suslin cardinal there is a hod pair $(\mathcal{P}, \Sigma)$ such that

1. the Wadge rank of Σ (or rather the set of reals coding Σ) is $\geq \kappa$ and

2. for some $\eta \in \mathcal{P}$, letting $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the direct limit of all countable Σ-iterates Q of $\mathcal{P}$ such that the iteration embedding $\pi^\Sigma_{\mathcal{P},Q}$ is defined and letting $\pi^\Sigma_{\mathcal{P},\infty} : \mathcal{P} \to \mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the iteration map, then V^{HOD}_κ is the universe of $\mathcal{M}_\infty(\mathcal{P}, \Sigma)|\pi^\Sigma_{\mathcal{P},\infty}(\eta)$[12].

A technical reformulation of the above fact appears as [7, Theorem 7.2.2].

The situation, however, is drastically different for the largest Suslin cardinal. Let κ be the largest Suslin cardinal. The inner model theoretic object that has Wadge rank κ cannot be an iteration strategy. This is because if Σ is an iteration strategy with nice properties like *hull condensation*[13] then assuming AD holds in $L(\Sigma, \mathbb{R})$, $L(\Sigma, \mathbb{R}) \vDash$ "$\mathcal{M}_1^{\sharp,\Sigma}$ exists and is ω_1-iterable"[14]. This then easily implies that Σ is both Suslin and co-Suslin. It then follows that no nice iteration strategy can have Wadge rank $\geq \kappa$, as any such strategy is both Suslin and co-Suslin[15].

The inner model theoretic object that has Wadge rank κ is a *short tree strategy*, which is a partial iteration strategy. Suppose $\mathcal{P}$ is any iterable structure and Σ is its iteration strategy. Suppose δ is a Woodin cardinal of $\mathcal{P}$. Given $\mathcal{T} \in dom(\Sigma)$ that is based on $\mathcal{P}|\delta$, we say that $\mathcal{T}$ is Σ-*short* if letting $\Sigma(\mathcal{T}) = b$, either the iteration map $\pi^\mathcal{T}_b$ is undefined or $\pi^\mathcal{T}_b(\delta) > \delta(\mathcal{T})$. If $\mathcal{T}$ is not Σ-short then we say that it is Σ-*maximal*. We then set Σ^{stc} be the fragment of Σ that acts on short trees.

Following [7, Definition 3.1.4] we make the following definition.

Definition 3.1 *Suppose $\mathcal{T}$ is a normal iteration tree of limit length. We then let*

$$\mathrm{m}(\mathcal{T}) = \cup_{\alpha<\mathrm{lh}(\mathcal{T})}\mathcal{M}^\mathcal{T}_\alpha|\mathrm{lh}(E^\mathcal{T}_\alpha) \textit{ and } \mathrm{m}^+(\mathcal{T}) = (\mathrm{m}(\mathcal{T}))^\#.$$

In the language of the above definition, the convention used in [7] is the following: $\Sigma^{stc}(\mathcal{T}) = b$ if and only if

1. $\mathcal{T}$ is Σ-short and $\Sigma(\mathcal{T}) = b$, or

[11]This theorem is probably due to Woodin. The outline of the proof is as follows. By an unpublished theorem of Woodin (but see [6, Theorem 1.9]), κ is a measurable cardinal, as it is a regular cardinal. It follows that there is an ω-club C consisting of members of the Solovay sequence such that for all $\lambda \in C$, $HOD \vDash$ "λ is regular". Hence, $L(\Gamma_\lambda, \mathbb{R}) \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \lambda = \Theta + \Theta$ is regular". For the proof of the last inference see [1, Theorem 2.3].

[12]Thus, $\pi^\Sigma_{\mathcal{P},\infty}(\eta) = \kappa$.

[13]Σ must also satisfy some form of generic interpretability, i.e., there must be a way to interpret Σ on the the generic extensions of $\mathcal{M}_1^{\sharp,\Sigma}$.

[14]This can be proved by a Σ^2_1-reflection argument.

[15]It follows from the theory of Suslin cardinals under AD that κ cannot be the largest Suslin cardinal, see [**?**, Chapter 3].

2. $\mathcal{T}$ is Σ-maximal and $b = \mathrm{m}^+(\mathcal{T})$.

Thus, Σ^{stc} tells us the branch of a Σ-short tree or the last model of a Σ-maximal tree.

The reader can perhaps imagine many ways of defining the notion of *short tree strategy* without a reference to an actual strategy. The convention that we adopt here is the following. If Λ is a short tree strategy for $\mathcal{P}$ then we will require that

1. for some $\mathcal{P}$-cardinal δ, $\mathcal{P} = (\mathcal{P}|\delta)^\#$ and $\mathcal{P} \models$ "δ is a Woodin cardinal",
2. if δ is as above and ν is the least $< \delta$-strong cardinal of $\mathcal{P}$ then $\mathcal{P} \models$ "ν is a limit of Woodin cardinals",
3. given an iteration tree $\mathcal{T} \in dom(\Lambda)$, $\Lambda(\mathcal{T})$ is either a cofinal well-founded branch of $\mathcal{T}$ or is equal to $\mathrm{m}^+(\mathcal{T})$,
4. for all iteration trees $\mathcal{T} \in dom(\Lambda)$, if $\Lambda(\mathcal{T})$ is a branch b then $\pi^{\mathcal{T}}_b(\delta) > \delta(\mathcal{T})$,
5. for all iteration trees $\mathcal{T} \in dom(\Lambda)$, if $\Lambda(\mathcal{T})$ is a model then $\mathrm{m}^+(\mathcal{T}) \models$ "$\delta(\mathcal{T})$ is a Woodin cardinal".

If a hod mouse $\mathcal{P}$ has properties 1 and 2 above then we say that $\mathcal{P}$ is of *#-lsa type*. [7, Definition 2.7.3] introduces other types of LSA hod premice.

The set of reals that has Wadge rank κ is some short tree strategy Λ. The hod mouse $\mathcal{P}$ that Λ iterates has a unique Woodin cardinal δ such that if $\nu < \delta$ is the least cardinal that is $< \delta$-strong in $\mathcal{P}$, then $\mathcal{P} \models$ "ν is a limit of Woodin cardinals". The aforementioned Woodin cardinal δ is also the largest Woodin cardinal of $\mathcal{P}$. This fact is proven in [7] (for example, see [7, Theorem 7.2.2] and [7, Chapter 8]). There is yet another way that the LSA stages of the Solovay Sequence are different from other points.

We continue assuming that V is the minimal model of LSA. If Σ is a strategy of a hod mouse with nice properties then ordinal definability with respect to Σ is captured by Σ-mice. More precisely, [7, Theorem 10.2.1] implies that if x and y are reals then x is ordinal definable from y using Σ as a parameter if and only if there is a Σ-mouse $\mathcal{M}$ over y[16] such that $x \in \mathcal{M}$.

[7, Theorem 10.2.1] also implies that the same conclusion is true for short tree strategies. Namely, if Λ is a short tree strategy then for x and y reals, x is ordinal definable from y using Λ as a parameter if and only if there is a Λ-mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$. Theorems of this sort are known as Mouse Capturing theorems. Such theorems are very important when analyzing models of determinacy using inner model theoretic tools.

For a strategy Σ the concept of a Σ-mouse has appeared in many places. The reader can consult [5, Definition 1.20] but the notion probably was first mentioned in [12] and was finally fully developed in [9].

A Σ-mouse $\mathcal{M}$, besides having an extender sequence also has a predicate that indexes the strategy. The idea, which is due to Woodin, is that the strategy predicate should index the branch of the least tree that has not yet been indexed.

Unfortunately this idea doesn't quite work for Λ-mice where Λ is a short tree strategy. In the next subsection, we will explain the solution presented in [7].

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[16] The difference between a mouse and a mouse over y is the same as the difference between L and $L[x]$.

3.1 Short tree strategy mice

We are assuming that V is the minimal model of LSA. Suppose Λ is a short tree strategy for a hod mouse $\mathcal{P}$. We let δ be the largest Woodin cardinal of $\mathcal{P}$. Thus, $\mathcal{P} = (\mathcal{P}|\delta)^{\#}$.

In general, when introducing any notion of a mouse one has to keep in mind the procedures that allow us to build such mice. Formally speaking, many notions of Λ-mice might make perfect sense, but when we factor into it the constructions that are supposed to produce such mice we run into a key issue.

In any construction that produces some sort of mouse (e.g. K^c-constructions, fully backgrounded constructions, etc) there are stages where one has to consider certain kinds of Skolem hulls, or as inner model theorists call them fine structural *cores*. The reader can view these cores as some carefuly defined Skolem hulls. To illustrate the aformentioned problem, imagine we do have some notion of Λ-mice and let us try to run a construction that will produce such mice. Suppose $\mathcal{T}$ is a tree according to Λ that appears in this construction. Having a notion of a Λ-mouse means that we have a prescription for deciding whether $\Lambda(\mathcal{T})$ should be indexed in the strategy predicate or not.

Suppose $\mathcal{T}$ is a Λ-maximal tree. It is hard to see exactly what one can index so that the strategy predicate remembers that $\mathcal{T}$ is maximal. And this "remembering" is the issue. Imagine that at a later stage we have a Skolem hull $\pi : \mathcal{M} \to \mathcal{N}$ of our current stage such that $\mathcal{T} \in rng(\pi)$. It is possible that $\mathcal{U} =_{def} \pi^{-1}(\mathcal{T})$ is Λ-short. If we have indexed X in our strategy that proves Λ-maximality of $\mathcal{T}$ then $\pi^{-1}(X)$ now can no longer prove that $\mathcal{U}$ is Λ-maximal. Thus, the notion of Λ-mouse cannot be first order.

The solution is simply not to index anything for Λ-maximal trees. This doesn't quite solve the problem as the above situation implies that nothing should be indexed for many Λ-short trees as well. To solve this problem, we will only index the branches of some Λ-short trees, those that we can locally prove are Λ-short. We explain this below in more details.

Fix an lsa type hod premouse $\mathcal{P}$ and let Λ be its short tree strategy. Let δ be the largest Woodin cardinal of $\mathcal{P}$ and ν be the least $< \delta$-strong of $\mathcal{P}$. To explain the exact prescription that we use to index Λ, we explain some properties of the models that have already been constructed according to this indexing scheme. Suppose $\mathcal{M}$ is a Λ-premouse.

Call $\mathcal{T} \in \mathcal{M}$ *universally short* (uvs) if $\mathcal{T}$ is obviously *short* (see [7, Definition 3.3.2]). For instance, it can be that the #-operator provides a $\mathcal{Q}$-structure and determines a branch c of $\mathcal{T}$ such that $\mathcal{Q}(c, \mathcal{T})$[17] exists and $\mathcal{Q}(c, \mathcal{T}) \trianglelefteq \mathrm{m}^+(\mathcal{T})$. Another way that a tree can be obviously short is that there could be a model $\mathcal{Q}$ in $\mathcal{T}$ such that $\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ is defined and the portion of $\mathcal{T}$ that comes after $\mathcal{Q}$ is based on $\mathcal{Q}^b$. Here $\mathcal{Q}^b$ is defined as $\mathcal{Q}|(\kappa^+)^{\mathcal{Q}}$, where κ is the supremum of the Woodin cardinals below the largest Woodin of $\mathcal{Q}$. The reader should keep in mind that there is a formula ζ in the language of Λ-premice such that for any Λ-premouse $\mathcal{M}$ and for any iteration tree $\mathcal{T} \in \mathcal{M}$, $\mathcal{T}$ is uvs if and only if $\mathcal{M} \vDash \zeta[\mathcal{T}]$.

Unfortunately there can be trees that are not universally short (nuvs). Suppose then $\mathcal{T}$ is nuvs. In this case whether we index $\Lambda(\mathcal{T})$ or not depends on whether we can find a $\mathcal{Q}$-structure that can be authenticated to be the correct one. There can be many ways to certify a $\mathcal{Q}$-structure, and [7] provides one such method. An interested reader can consult [7, Section 3.7]. Notice that because $\mathcal{P}$ has only one Woodin cardinal, not being able to find a $\mathcal{Q}$-structure is equivalent to the tree being maximal. Thus, in a nutshell the solution proposed by [7] is that we index only branches that are given by *internally* authenticated $\mathcal{Q}$-structures.

Suppose now that we have the above Skolem hull situation, namely that we have $\pi : \mathcal{M} \to \mathcal{N}$ and $\mathcal{T}$ in $\mathcal{N}$ that is Λ-maximal but $\pi^{-1}(\mathcal{T})$ is short. There is no more indexing problem. The reason is

[17] $\mathcal{M}^{\mathcal{T}}_c$ is a direct limit along the models of c. $\mathcal{Q}(c, \mathcal{T})$ is the largest initial segment of $\mathcal{M}^{\mathcal{T}}_c$ such that $\mathcal{Q}(c, \mathcal{T}) \vDash$ "$(\mathcal{T})$ is a Woodin cardinal". It is only defined provided that $(\mathcal{T})$ is not a Woodin cardinal for some function definable over $\mathcal{M}^{\mathcal{T}}_c$.

that in order to index $\Lambda(\pi^{-1}(\mathcal{T}))$ in $\mathcal{M}$ we need to find an authenticated Q-structure for $\pi^{-1}(\mathcal{T})$. The authentication process is first order, and so if $\mathcal{N}$ does not have such an authenticated Q-structure for $\mathcal{T}$ then $\mathcal{M}$ cannot have such an authenticated Q-structure for $\pi^{-1}(\mathcal{T})$.

The authentication procedure is internal to the mouse. More precisely, the following holds:

Internal Definability of Authentication: there is a formula ϕ in the appropriate language such that whenever $(\mathcal{P}, \Lambda)$ is as above and $\mathcal{M}$ is a Λ-mouse over some set X such that $\mathcal{P} \in X$, for any iteration tree $\mathcal{T} \in \mathcal{M}$, $\mathcal{M} \vDash \phi[\mathcal{T}]$ if and only if $\mathcal{T} \in dom(\Lambda)$, $\mathcal{T}$ is short and $\Lambda(\mathcal{T}) \in \mathcal{M}$.

We again note that the Internal Definability of Authentication (IDA) is only shown to be true for the minimal model of LSA. In general, IDA cannot be true as there can be short trees without Q-structures. The authors have recently discovered another short tree indexing scheme that can work in all cases, but has some weaknesses compared to the one introduced in [7].

Using the notation in [7], recall that $\mathcal{P}^b$ is the "bottom part" of $\mathcal{P}$, i.e $\mathcal{P}^b = \mathcal{P}|(\nu^+)^{\mathcal{P}}$, where ν is the supremum of the Woodin cardinals below the top Woodin of $\mathcal{P}$.

We now describe another key feature of the indexing scheme of [7] that is of importance here. We say Σ is a *low level component* of Λ if there is a tree $\mathcal{T}$ on $\mathcal{P}$ according to Λ such that $\pi^{\mathcal{T},b}$ exists[18] ($\mathcal{T}$ may be $\emptyset$) and for some $\mathcal{R} \trianglelefteq \pi^{\mathcal{T},b}(\mathcal{P}^b)$, $\Sigma = \Lambda_{\mathcal{R}}$. Let $LLC(\Lambda)$ be the set of Σ that are a low level components of Λ. What is shown in [7] is that Λ is determined by $LLC(\Lambda)$ in a strong sense.

Given a transitive model M of a fragment of ZFC such that $\mathcal{P} \in M$ we say M is closed under $LLC(\Lambda)$ if whenever $\mathcal{T} \in M$ is a tree according to Λ such that $\pi^{\mathcal{T},b}$ exists, $\Lambda_{\pi^{\mathcal{T},b}(\mathcal{P}^b)}$ has a universally Baire representation over M. More precisely, whenever $g \subseteq Coll(\omega, \pi^{\mathcal{T},b}(\mathcal{P}^b))$ is M-generic, for every M-cardinal λ there are trees $T, S \in M[g]$ on λ such that $M[g] \vDash$ "(T, S) are $< \lambda$-complementing" and for all $< \lambda$-generics h, $(p[T])^{M[g*h]} = Code(\Lambda_{\pi^{\mathcal{T},b}(\mathcal{P}^b)}) \cap M[g * h]$. Here $Code(\Phi)$ is the set of reals coding Φ (with respect to a fixed coding of elements of HC by reals).

It is shown in [7] that if, assuming AD^+, (M, Σ) is such that

1. M is a countable model of a fragment of ZFC,
2. M has a class of Woodin cardinals,
3. Σ is an ω_1-iteration strategy for M and
4. whenever $i : M \to N$ is an iteration via Σ, N is closed under $LLC(\Lambda)$,

then there is a formula ψ such that whenever g is M-generic, for any $\mathcal{T} \in M[g]$,

$$\mathcal{T} \text{ is according to } \Lambda \text{ if and only if } M[g] \vDash \psi[\mathcal{T}]. \qquad (\star)$$

The interested reader can consult Chapters 5, 6 and 8 of [7].

The reason we explained the above is to give the reader some confidence that defining a short tree strategy Λ for a hod premose $\mathcal{P}$ is equivalent to describing the set $LLC(\Lambda)$. This fact is the reason that the indexing schema of [7] works in the following sense.

Being able to define short-tree-strategy mice is one thing, proving that they are useful is another. Usually what needs to be shown are the following two key statements. We let ϕ_{sts} be the formula that is mentioned in the Internal Definability of Authentication.

[18] $\pi^{\mathcal{T},b}$ is the restriction of the iteration embedding to $\mathcal{P}^b$. See [7], just after Definition 2.7.21, for a more detailed definition. Note that in some cases, $\pi^{\mathcal{T},b}$ may exist but $\pi^{\mathcal{T}}$ may not.

The Eventual Authentication. Suppose $(\mathcal{P}, \Lambda)$ is as above and $\mathcal{M}$ is a sound Λ-mouse over some set X such that $\mathcal{P} \in X$ and $\mathcal{M}$ projects to X. Suppose $\mathcal{T} \in \mathcal{M}$ is according to Λ and is Λ-short. Suppose further that $\mathcal{M} \vDash \neg\phi_{sts}[\mathcal{T}]$. Then there is a sound Λ-mouse $\mathcal{N}$ over X such that $\mathcal{M} \trianglelefteq \mathcal{N}$ and $\mathcal{N} \vDash \phi_{sts}[\mathcal{T}]$.[19]

Mouse Capturing for Λ**:** Suppose $(\mathcal{P}, \Lambda)$ is as above. Then for any $x \in \mathbb{R}$ that codes $\mathcal{P}$ and any $y \in \mathbb{R}$, y is ordinal definable from x and Λ if and only if there is a Λ-mouse $\mathcal{M}$ over x such that $y \in \mathcal{M}$.

Both The Eventual Authentication and Mouse Capturing for Λ are proven in [7] (see [7, Chapter 8, Lemma 8.1.3, Lemma 8.1.5] and [7, Theorem 10.2.1]).

The next subsection discusses the Q-structure authentication process mentioned above.

3.2 The authentication method

Suppose $\mathcal{P}$ is a #-lsa type hod premouse. Recall from the previous subsections that this means that $\mathcal{P}$ has a largest Woodin cardinal δ such that $\mathcal{P} = (\mathcal{P}|\delta)^{\#}$ and the least $< \delta$-strong cardinal of $\mathcal{P}$ is a limit of Woodin cardinals. We let $\delta^{\mathcal{P}}$ be the largest Woodin cardinal of $\mathcal{P}$ and $\kappa^{\mathcal{P}}$ be the least $< \delta^{\mathcal{P}}$-strong cardinal of $\mathcal{P}$. We shall also require that $\mathcal{P}$ is *tame*, meaning that for any $\nu < \delta^{\mathcal{P}}$, if $(\mathcal{P}|\nu)^{\#}$ is of lsa type and $\mathcal{M} \triangleleft \mathcal{P}$ is the largest such that $\mathcal{M} \vDash$ "ν is a Woodin cardinal" then ν is not overlapped in $\mathcal{M}$[20].

Our goal here is to explain the Q-structure authentication procedure employed by [7]. Recall our discussion of uvs and nuvs trees. The Q-structure authentication procedure applies to only nuvs trees, trees that are not obviously short.

[7, Chapters 3.6-3.9] develop the aforementioned authentication procedure. [7, Definition 3.8.9, 3.8.16, 3.8.17] introduce the sts indexing scheme. For illustrative purposes, it is better to think of the indexing scheme introduced there as a hierarchy of indexing schemes indexed by ordinals. Naturally, this hierarchy is defined by induction. For illustrative purposes we call γth level of the hierarchy sts_γ. Thus, $sts_\gamma(\mathcal{P})$ is the set of all sts premice that are based on $\mathcal{P}$ (i.e., their short tree strategy predicate describes a short tree strategy for $\mathcal{P}$) and have rank $\leq \gamma$.

To begin the induction, we let $sts_0(\mathcal{P})$ be the set of all sts premice that do not index a branch for any nuvs tree. More precisely, if $\mathcal{M} \in sts_0(\mathcal{P})$ and $\mathcal{T} \in dom(S^{\mathcal{M}})$ then if $S^{\mathcal{M}}(\mathcal{T})$ is defined then $\mathcal{T}$ is uvs.

Below and elsewhere, $S^{\mathcal{M}}$ is the strategy predicate of $\mathcal{M}$. Given $sts_\alpha(\mathcal{P})$ we let $sts_{\alpha+1}(\mathcal{P})$ be the set of all sts premice that index branches of those nuvs trees that have a Q-structure in $sts_\alpha(\mathcal{P})$. More precisely, suppose $\mathcal{M} \in sts_{\alpha+1}(\mathcal{P})$ and $\mathcal{T} \in dom(S^{\mathcal{M}})$ and $S^{\mathcal{M}}(\mathcal{T})$ is defined. Then either

1. $\mathcal{T}$ is uvs or
2. $\mathcal{T}$ is nuvs and there is $Q \in \mathcal{M}$ such that $\mathcal{M} \vDash$ "$Q \in sts_\alpha(\mathcal{P})$", $\mathrm{m}^+(\mathcal{T}) \triangleleft Q$, $Q \vDash$ "$\delta(\mathcal{T})$ is a Woodin cardinal" but $\delta(\mathcal{T})$ is not a Woodin cardinal with respect to some function definable over Q[21] and there is a cofinal branch b of $\mathcal{T}$ such that $Q \triangleleft \mathcal{M}_b^{\mathcal{T}}$.

When Q exhibits the properties listed in clause 2 we say that Q is a Q-structure for $\mathcal{T}$. It follows from the zipper argument of [3, Theorem 2.2] that for each Q-structure Q there is at most one branch b with properties described in clause 2 above. However, there is nothing that we have said so far that guarantees the uniqueness of the Q-structure itself. The uniqueness is usually a consequence of iterability and

[19] One can then prove that there is such an $\mathcal{N}$ that projects to X.

[20] This means that if $E \in \vec{E}^{\mathcal{M}}$ then $\nu \notin (\mathrm{crit}(E), index(E))$.

[21] This can be written as $\mathcal{J}_1(Q) \vDash$ "$\delta(\mathcal{T})$ is not a Woodin cardinal".

comparison (see [15, Theorem 3.11])[22]. Thus, to make the definition of $sts_{\alpha+1}$ complete, we need to impose an iterability condition on Q.

The exact iterability condition that one needs is stated as clause 5 of [7, Definition 3.8.9]. This clause may seem technical, but there are good reasons for it. For the purposes of identifying a unique branch b saying that Q in clause 2 is sufficiently iterable in $\mathcal{M}$ would have sufficed. However, recall the statement of the Internal Definability of Authentication. The problem is that when we require that an $\mathcal{M}$ as above is a Λ-premouse we in addition must say that the branch b that the Q-structure Q defines is the exact same branch that Λ picks. To guarantee this, we need to impose a condition on Q such that Q will be iterable not just in $\mathcal{M}$ but in V. The easiest way of doing this is to say that Q has an iteration strategy in some derived model as then, using genericity iterations (see [15, Chapter 7.2]), we can extend such a strategy for Q to a strategy that acts on iterations in V.

For limit α, $sts_\alpha(\mathcal{P})$ is essentially $\bigcup_{\beta<\alpha} sts_\beta(\mathcal{P})$. What has been left unexplained is the kind of strategy that the Q-structure Q must have in some derived model. Let Σ be this strategy. If $\mathcal{M} \in sts_\alpha(\mathcal{P})$ is a Λ-mouse then Q must be a $\Lambda_{\mathrm{m}^+(\mathcal{T})}$-mouse over $\mathrm{m}^+(\mathcal{T})$. Thus, our next challenge is to find a first order way of guaranteeing that Σ-iterates of Q are $\Lambda_{\mathrm{m}^+(\mathcal{T})}$-mice, even those iterates that we will obtain after blowing up Σ via genericity iterations.

The solution that is employed in [7] is that if $\mathcal{R}$ is a Σ-iterate of Q and $\mathcal{U} \in dom(S^{\mathcal{R}})$ then $\mathcal{U}$ itself is authenticated by the extenders of $\mathcal{M}$. Below we refer to this certification as *tree certification*. This is again a rather technical notion, but the following essentially illustrates the situation.

Let us suppose $\mathcal{R} = Q$ and $\mathcal{U} \in dom(S^{Q})$. The indexing scheme of [7] does not index all trees in $\mathcal{P}$. In other words, $S^{\mathcal{M}}$ is never total. $dom(S^{\mathcal{M}})$ consists of trees that are built via a comparison procedure that iterates $\mathcal{P}$ to a background construction of $\mathcal{M}$. Set $\mathcal{N} = \mathrm{m}^+(\mathcal{T})$. One requirement is that $\mathcal{N}$ also iterates to one such background construction to which $\mathcal{P}$ also iterates. Let $\mathcal{S}$ be this common background construction and suppose $\alpha+1 < lh(\mathcal{U})$ is such that α is a limit ordinal. First assume $\mathcal{U} \restriction \alpha$ is uvs. What is shown in [7] is that knowing the branch of $\mathcal{P}$-to-$\mathcal{S}$ tree there is a first order procedure that identifies the branch of $\mathcal{U} \restriction \alpha$, and that procedure is the tree certification procedure applied to $\mathcal{U} \restriction \alpha$.

Suppose next that $\mathcal{U} \restriction \alpha$ is nuvs. Then because $\alpha + 1 < lh(\mathcal{U})$, $\mathcal{U} \restriction \alpha$ must be short and the branch chosen for it in Q must have a Q-structure Q_1 which is itself an sts mouse. We have that $Q_1 \in Q$ and Q_1 must have the same certification in Q that Q has in $\mathcal{M}$. Again, the nuvs trees in Q_1 have a tree certification in Q according to the above procedure. The uvs ones produce another $Q_2 \in Q_1$. Because we cannot have an infinite descent, the definition of tree certification is meaningful.

Remark 3.2 *It is sometimes convenient to think of a short tree strategy as one having two components, the branch component and the model component. Given a short tree strategy Λ, we let $b(\Lambda)$ be the set of those trees $\mathcal{T} \in dom(\Lambda)$ such that $\Lambda(\mathcal{T})$ is a branch of $\mathcal{T}$, and we let $m(\Lambda)$ be the set of those trees $\mathcal{T} \in dom(\Lambda)$ such that $\Lambda(\mathcal{T})$ is a model.*

The convention adopted here is that if $\mathcal{T} \in m(\Lambda)$ then $\Lambda(\mathcal{T}) = \mathrm{m}^+(\mathcal{T})$[23]. Thus, if $\mathcal{M}$ is an sts premouse then $S^{\mathcal{M}}$ is a short tree strategy in the above sense, i.e., for $\mathcal{T} \notin b(S^{\mathcal{M}})$, $S^{\mathcal{M}}(\mathcal{T})$ is simply left undefined.

[22] In general, the theory of Q-structures doesn't have much to do with sts mice. It will help if the reader develops some understanding of [15, Chapter 6.2 and Definition 6.11].

[23] It is not up to us to decide whether $\Lambda(\mathcal{T}) \in m(\Lambda)$ or $\Lambda(\mathcal{T}) \in b(\Lambda)$. The short-tree strategy itself decides this.

4 Proof of Theorem 1.2

We assume $(\mathcal{P}, \Sigma)$ is a hod pair or a sts hod pair. Recall the terminology of meek, etc associated with hod premice from [7].

1. (Meek) There is δ such that

   (a) $\mathcal{P} \models$ "δ is a Woodin cardinal or a limit of Woodin cardinals",

   (b) δ is a cutpoint of $\mathcal{P}$[24],

   (c) if $\kappa < \mathrm{ord}(\mathcal{P})$ is a limit of Woodin cardinals of $\mathcal{P}$ then $o^{\mathcal{P}}(\kappa) < \delta$,

   (d) $\mathcal{P} \models \mathsf{ZFC} - \mathsf{Replacement}$ and

   (e) if δ is a Woodin cardinal of $\mathcal{P}$ then $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}|(\delta^{+n})^{\mathcal{P}}$, and if δ is a limit of Woodin cardinals of $\mathcal{P}$ then δ is the largest cardinal of $\mathcal{P}$.

2. (Non-meek) There is $\delta \leq \mathrm{ord}(\mathcal{P})$ such that

   (a) there is $\kappa < \delta$ such that $\delta \leq o^{\mathcal{P}}(\kappa)$,

   (b) if κ is the least $\eta < \delta$ such that $\delta \leq o^{\mathcal{P}}(\eta)$ then $o^{\mathcal{P}}(\kappa) = \delta$ and $\mathcal{P} \models$ "κ is a limit of Woodin cardinals",

   (c) letting $\kappa < \delta$ be the least such that $o^{\mathcal{P}}(\kappa) = \delta$, $\rho(\mathcal{P}) \in (\kappa, \delta]$ or $\mathrm{ord}(\mathcal{P})$ is a limit of ordinals ξ such that $\rho(\mathcal{P}||(\xi, \omega)) \in (\kappa, \delta]$.

   (d) $\mathcal{P}$ is δ-sound,

   (e) if $\mathrm{dom}(\vec{E}^{\mathcal{P}}) \cap (\delta^{\mathcal{P}}, \mathrm{ord}(\mathcal{P})] = \emptyset$ then $\mathcal{J}_\omega[\mathcal{P}] \models$ "$\delta^{\mathcal{P}}$ is not a Woodin cardinal".

3. (Gentle) $\delta =_{def} \mathrm{ord}(\mathcal{P})$ is a limit of Woodin cardinals of $\mathcal{P}$ and $\mathcal{P} \models \mathsf{ZFC} - \mathsf{Replacement}$.

We let $\delta^{\mathcal{P}}$ be the δ above. We say $\mathcal{P}$ is of **lsa type** if

1. $\mathcal{P}$ is properly non-meek,

2. $\mathcal{P} \models$ "$\delta^{\mathcal{P}}$ is a Woodin cardinal"

Let $\alpha = \min(\mathrm{dom}(\vec{E}^{\mathcal{P}}) - \delta^{\mathcal{P}})$ (if exists). We then say that $\mathcal{P}$ is of **#-lsa type** if $\mathcal{P}|\alpha = \mathcal{P}$ and $\mathcal{J}_\omega[\mathcal{P}] \models$ "$\delta^{\mathcal{P}}$ is a Woodin cardinal". We say $\mathcal{P}$ is of successor type if $\mathcal{P}$ is meek and $\delta = \delta^{\mathcal{P}}$ is a Woodin cardinal in $\mathcal{P}$.

We now recall fullness preservation and branch condensation.

Definition 4.1 (Γ-Fullness preservation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair*[25] *such that $\mathcal{P} \in \mathsf{HC}$ and Γ is a projectively closed pointclass. We say Σ is **Γ-fullness preserving** if the following holds for all $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$.*

1. *For all meek layers $\mathcal{R}$ of $\mathcal{Q}$ such that $\mathcal{R}$ is of successor type, letting $\mathcal{S} = \mathcal{R}^{-}$*[26]*, for all $\eta \in (\mathrm{ord}(\mathcal{S}), \mathrm{ord}(\mathcal{R}))$ if η is a cutpoint cardinal of $\mathcal{R}$ then*

$$(\mathcal{R}|(\eta^+)^{\mathcal{R}})^* = \mathsf{Lp}^{\Sigma_{\mathcal{S},\mathcal{T}}}(\mathcal{R}|\delta).$$

2. *For all meek layers $\mathcal{R}$ of $\mathcal{Q}$ such that $\mathcal{R}$ is of limit type,*

[24] This condition follows from the other conditions, but we would like to isolate it.

[25] Recall that if $(\mathcal{P}, \Sigma)$ is an sts hod pair then $\mathcal{P} = (\mathcal{P}|\delta^{\mathcal{P}})^{\#}$. See Definition **??**.

[26] This is the longest proper layer of $\mathcal{R}$.

$$\mathcal{R} = \mathsf{Lp}^{\Sigma_{\mathcal{R}|\delta^{\mathcal{R}},\mathcal{T}}}(\mathcal{R}|\delta^{\mathcal{R}}).$$

3. *If $\mathcal{P}$ is of #-lsa type then $\mathsf{Lp}^{\Gamma,\Sigma^{stc}_{Q,\mathcal{T}}}(Q) \models$ "Q is a Woodin cardinal"*[27].

Definition 4.2 (Strongly Γ-fullness preserving) *Suppose $(\mathcal{P},\Sigma)$ is a hod pair or an sts hod pair and Γ is a pointclass. We say Σ is strongly Γ-fullness preserving if Σ is Γ-fullness preserving and whenever*

1. *$\mathcal{T}$ is a stack according to Σ with last model $\mathcal{S}$ such that if $\mathcal{P}$ is of limit type then $\pi^{\mathcal{T},b}$ exists and otherwise $\pi^{\mathcal{T}}$ exists, and*
2. *$\mathcal{R}$ is such that there are elementary embedding (σ,τ) with the property that*
   (a) *if $\mathcal{P}$ is of limit type then $\sigma : \mathcal{P}^b \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}^b$ and $\pi^{\mathcal{T},b} = \tau\circ\sigma$, and*
   (b) *if $\mathcal{P}$ is of successor type then $\sigma : \mathcal{P} \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}$ and $\pi^{\mathcal{T}} = \tau\circ\sigma$,*

then the τ-pullback strategy of $\Sigma_{\mathcal{S}^b,\mathcal{T}}$ if 2(a) holds and of $\Sigma_{\mathcal{S},\mathcal{T}}$ if 2(b) holds is Γ-fullness preserving. Following Definition 4.1 we can also define the meaning of ***strongly almost Γ-fullness preserving*** *as well as the meaning of* ***strongly low-level Γ-fullness preserving****.* ⊣

The definition of branch condensation for strategies is standard and can be found in [5]. We define branch condensation for st strategies.

Definition 4.3 (Branch condensation for st-strategies) *Suppose $(\mathcal{P},\Sigma)$ is such that $\mathcal{P}$ is a hod-like #-lsa type* lses *and Σ is a st-strategy for $\mathcal{P}$. We say Σ has* ***branch condensation*** *if whenever $(\mathcal{T},Q,\mathcal{U},\mathcal{R},\tau,\mathcal{S},c,\alpha,b)$ is such that*

1. *$(\mathcal{T},Q),(\mathcal{U},\mathcal{R}) \in I^b(\mathcal{P},\Sigma)$,*
2. *$\alpha < \lambda^{\mathcal{R}^b}$ and $\delta^{\mathcal{R}(\alpha+1)}$ is a Woodin cardinal of $\mathcal{R}$,*
3. *$\mathcal{S}$ is a normal iteration tree of limit length according to $\Sigma_{\mathcal{R}^b,\mathcal{U}}$ that is based on $\mathcal{R}(\alpha+1)$ and is above $\delta^{\mathcal{R}}_\alpha$,*
4. *c is a branch of $\mathcal{S}$ such that $\pi^{\mathcal{S}}_c$ exists, and*
5. *$\tau : \mathcal{M}^{\mathcal{S}}_c \to Q(b)$ and $\pi^{\mathcal{T},b} = \tau\circ\pi^{\mathcal{S}}_c\circ\pi^{\mathcal{U},b}$*

then $c = \Sigma_{\mathcal{R},\mathcal{U}}(\mathcal{S})$. ⊣

We first prove:

Lemma 4.4 *$L(\Sigma,\mathbb{R}) \models \mathsf{MC}(\Sigma)$.*

Proof. For simplicity, we assume $\Sigma = \emptyset$. The general is only more notationally more complicated. Suppose $y \in OD(x)$. Let β be the least such that $y \in OD^\beta(x)$ i.e. there is a formula φ and some $\vec{\beta} \in \alpha^{<\omega}$ such that for all $\bar{x}$,

$$\bar{x} = x \Leftrightarrow J_\beta(\mathbb{R}) \models \varphi[\bar{x},\vec{\beta},x].$$

By minimality of β, β is a critical ordinal (cf. [13]). Suppose k is the least such that $\rho_k(J_\beta(\mathbb{R})) = \mathbb{R}$, then $\Sigma^{J_\beta(\mathbb{R})}_{2n+k}$ and $\Pi^{J_\beta(\mathbb{R})}_{2n+k+1}$ have the scale property. Let $\Gamma = \Sigma^{J_\beta(\mathbb{R})}_{2n+k}$ for some sufficiently large n such that $y \in OD(x)$ as witnessed by a Γ formula.

[27] Here, if Σ is a short tree strategy then $\Sigma^{stc} = \Sigma$.

Claim 4.5 *There is a Turing cone of real z such that if $u \in OD^{\beta}(z)$ as witnessed by a Γ-formula then there is a z-mouse $\mathcal{M}$ such that $u \in \mathcal{M}$ and $\mathcal{M}$ has an ω_1-iteration strategy in $J_{\beta+1}(\mathbb{R})$. In fact, the operator $z \mapsto C_\Gamma(z)$ is fine-structural as witnessed in $J_{\beta+1}(\mathbb{R})$.*

Proof. It suffices, by Rudominer-Steel (cf.[10]), to show for any real y, there is $y \leq_T x$ and an x-mouse $\mathcal{R}$ such that $C_\Gamma(x) \subsetneq \mathbb{R} \cap \mathcal{R}$ and $\mathcal{R}$ is ω_1-iterable in $J_{\beta+1}(\mathbb{R})$.

Let A be a universal Γ-set and (M, δ, Σ) be a Γ-Woodin mouse that Suslin captures A (as witnessed by T, S) and that $\Sigma \in J_{\beta+1}(\mathbb{R})$ and $M \vDash$ "δ is Woodin." Let $(\mathcal{N}_\eta : \eta \leq \delta)$ be the models of the $L[E, y]$-construction of M. Let $\pi : M^* \to M$ be such that $S, T \in \text{ran}(\pi)$, $\text{ran}(\pi) \cap \delta = \gamma \in \delta$, and γ is not Woodin in $\mathcal{N}_\delta =_{def} Q$. Because $S, T \in \text{ran}(\pi)$, $C_\Gamma(Q|\gamma) \subset M^*$ and hence η is Woodin with respect to all $A \in C_\Gamma(Q|\gamma)$. Let $\xi > \gamma$ be the least such that there is a subset of γ in $Q|\gamma + 1 =_{def} \mathcal{P}$ but not in $C_\Gamma(Q|\gamma)$.

Let $x \in \mathbb{R}$ code a generic $g \subset Col(\omega, Q|\gamma)$. So $y \leq_T x$ and there is a $b_x \in \mathcal{P}[x]$ such that $b_x \notin C_\Gamma(x)$. $\mathcal{P}[x]$ can be re-arranged into an x-mouse $\mathcal{R}$. Since $\mathcal{R}$'s iteration strategy can be computed from $\mathcal{P}$'s strategy, $\mathcal{R}$ is iterable in $J_{\beta+1}(\mathbb{R})$. This proves the claim. □

Since $y \in C_\Gamma(x)$, we want to show there is an x-mouse $\mathcal{M}$ iterable in $J_{\beta+1}(\mathbb{R})$ such that $y \in \mathcal{M}$. Let $B \in \Gamma$ and $\xi < \omega_1$ be such that y is unique with $(y, z, x) \in B$ for any z coding ξ. Since Γ has the scale property

$$(C_\Gamma(x), B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}, B).$$

We may also assume B codes the $(2n+k)^{th}$-reduct of $J_\beta(\mathbb{R})$, and therefore, the fact that $(C_\Gamma(x), B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}, B)$ gives a Σ_{2n+k}-elementary embedding

$$\pi : J_{\bar{\beta}}(C_\Gamma(x)) \to J_\beta(\mathbb{R})$$

for some $\bar{\beta} \leq \beta$. The above fact holds for any real z, not just x.

By the claim and its proof, fix z^* such that for any $z^* \leq_T z$, there is a z-mouse in $J_{\beta+1}(\mathbb{R})$ whose reals are those in $C_\Gamma(z)$. Let $\Omega = \Sigma^{J_\beta(\mathbb{R})}_{2n+k+2}$ and (M, δ, Σ) be a coarse Ω-Woodin mouse in $J_{\beta+1}(\mathbb{R})$ such that $z^*, x \in M$. Let $(\mathcal{N}_\eta : \eta \leq \delta)$ be the models of the $L[E, x]$-construction in M. Let $Q = \mathcal{N}_\delta$ and as above, fix $\gamma < \xi < \delta$ such that $Q|\xi$ projects to γ and defines a subset of γ not in $C_\Gamma(Q|\gamma \cup \{x, z^*\})$. Note that $\mathcal{P} = Q|\xi \vDash$ "γ is Woodin" and also that $\{x, z^*\}$ is $\mathbb{P}$-generic over $\mathcal{P}$, where $\mathbb{P}$ is the extender algebra defined in $\mathcal{P}$.

Let z code a $Col(\omega, Q|\gamma \cup \{x, z^*)$-generic. We can re-arrange $\mathcal{P}[z]$ into a z-mouse $\mathcal{R}$ as before. Note that

$$C_\Gamma(z) = \mathbb{R} \cap \mathcal{R}.$$

Since $x \leq_T z$, $y \in \mathcal{R}$. Furthermore, y is definable over $J_{\bar{\beta}}(C_\Gamma(z))$ for some countable $\bar{\beta}$ and the least such model is in $\mathcal{R}$. So y is OD in $\mathcal{R}$. Therefore, $y \in Q$. □

Now we prove the theorem. Let $\mathcal{R} \trianglelefteq^c_{hod} \mathcal{P}$ be the least layer of $\mathcal{P}$. We want to show that

(1) $L(\Sigma, \mathbb{R}) \vDash$ "Mouse Capturing holds for $\Sigma_{\mathcal{R}}$".

The general case is only notationally more complex. Suppose $x, y \in \mathbb{R}$ are such that $L(\Sigma, \mathbb{R}) \vDash$ "$y \in OD_{\Sigma_{\mathcal{R}}}(x)$". It follows from Theorem 1.2 that there is a Σ-mouse $\mathcal{M}$ over $(\mathcal{P}, x)$ containing y such that $\mathcal{M}$ has an iteration strategy in $L(\Sigma, \mathbb{R})$. In fact, it follows from Theorem 1.2 that

(2) for every $Q \in pI(\mathcal{P}, \Sigma)$ there is a Σ_Q-mouse $\mathcal{M}$ over (Q, x) such that $y \in \mathcal{M}$ and $\mathcal{M}$ has an iteration strategy in $L(\Sigma, \mathbb{R})$.[28]

[28] This is because $L(\Sigma_Q, \mathbb{R}) = L(\Sigma, \mathbb{R})$ and $L(\Sigma_Q, \mathbb{R}) \vDash \mathsf{MC}(\Sigma_Q)$.

Let $\mathcal{M}_Q$ be the least Σ_Q-mouse over (Q, x) such that y is definable over $\mathcal{M}_Q$. Let Λ_Q be the iteration strategy of $\mathcal{M}_Q$ (witnessing that $\mathcal{M}_Q$ is a Σ_Q-mouse). Let $\Gamma^* \in L(\Sigma, \mathbb{R})$ be a good pointclass such that the set

$$A = \{(z, u) \in \mathbb{R}^2 : z \text{ codes some } \mathcal{M}_Q \text{ and } u \text{ is an iteration according to } \Lambda_Q\}$$

is in Δ_{Γ^*}. Let F be as in Theorem 2.7 for Γ^* and let $z \in \mathrm{dom}(F)$ be such that if $F(z) = (\mathcal{N}_z^*, \mathcal{M}_z, \delta_z, \Sigma_z)$ then $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures Σ and the set A. Let $\mathcal{N} = (\mathsf{Le}(\emptyset, x))^{\mathcal{N}_z^*|\delta_z}$. It follows from Theorem 1.3 that

(3) there is a $Q \in \mathcal{N}$ such that $\Sigma_Q \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.

It follows from the universality of $\mathcal{N}$ that $\mathcal{M}_Q \in \mathcal{N}$ (this is because $(\mathsf{Le}((Q, \Sigma_Q))^{\mathcal{N}}$ is universal in $\mathcal{N}_z^*$ and the strategy Λ_Q of $\mathcal{M}_Q$ is captured by $\mathcal{N}_z^*$ (via A)). It then follows that $y \in \mathcal{N}$. As $\mathcal{N}$ is an x-mouse, this completes the proof.

Remark 4.6 *The case $(\mathcal{P}, \Sigma)$ is anomalous is handled as in [5, Lemma 6.22]. The main issue is we cannot use Theorem 1.3 as (Q, Σ_Q) need not be fullness preserving.*

5 Proof of Lemma 1.4

Given a set of reals $A \subseteq \mathbb{R}$, we let $W_A = \{B \subseteq \mathbb{R} : w(B) < w(A)\}$. Next following Definition 3.13 of [4], we say $A \subseteq \mathbb{R}$ is a new set if

1. $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$,
2. $\wp(\mathbb{R}) \cap L(W_A, \mathbb{R}) = W_A$,
3. $\Theta^{L(W_A, \mathbb{R})}$ is a Suslin cardinal of $L(A, \mathbb{R})$.

The following is [4, Definition 3.17].

Definition 5.1 *Given a pointclass Γ, we say Γ is **completely mouse full** if either $\Gamma = \wp(\mathbb{R})$ or there is a new set A such that*

1. *$\Gamma = W_A$,*
2. *if $(\mathcal{P}, \Sigma)$ is allowable (see [7, Definition 3.10.7] such that $\mathsf{Code}(\Sigma) \in \Gamma$ and $L(A, \mathbb{R}) \vDash$ "Σ has strong branch condensation and is Γ-fullness preserving" then for every $a \in HC$,*

$$Lp^{\Gamma,\Sigma}(a) = (Lp^{\Sigma}(a))^{L(A,\mathbb{R})}.$$

$\dashv$

Given two pointclasses Γ_1 and Γ_2, we write $\Gamma_1 \lhd_{mouse} \Gamma_2$ if $\Gamma_1 \subseteq \Gamma_2$ and Γ_2 has the same mice as Γ_1 relative to common iteration strategies. More precisely, if $(\mathcal{P}, \Sigma) \in \Gamma_1$ is an allowable pair such that $L(\Gamma_2, \mathbb{R}) \vDash$ "Σ has strong branch condensation and is Γ_1-fullness preserving" then for any $a \in HC$,

$$Lp^{\Gamma_1,\Sigma}(a) = Lp^{\Gamma_2,\Sigma}(a).$$

Finally, following [4, Definition 3.18],

Definition 5.2 Γ *is **mouse full** if either it is completely mouse full or is a union of completely mouse full pointclasses* $(\Gamma_\alpha : \alpha < \Omega^\Gamma)$ *such that for all* α, $\Gamma_\alpha \lhd_{mouse} \Gamma_{\alpha+1}$ *and for all limit* α, $\Gamma_\alpha = \bigcup_{\beta<\alpha}\Gamma_\beta$.

In this subsection we outline the proof of Lemma 1.4. Suppose that there is no hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that

1. Σ has strong branch condensation and is strongly fullness preserving,
2. $\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R})$

The following theorem is the key "Generation of Mousefull Pointclass" Theorem ([5, Theorem 6.1] and [7, 10.1.2]). We operate under the assumption that there is no Γ such that $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$ but there is a Suslin cardinal above $w(\Gamma)$.

Theorem 5.3 ($\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$) *Suppose* $\Gamma \subsetneq \wp(\mathbb{R})$ *is a mousefull pointclass such that* $\Gamma \vDash \mathsf{SMC}$. *Then there is a hod pair or sts hod pair or an anomalous pair* $(\mathcal{P}, \Sigma)$ *that generates* Γ.[29]

Lemma 5.4 *Suppose* $(\mathcal{P}, \Sigma) \in \Gamma$ *is a hod pair such that* Σ *has strong branch condensation and being super fullness preserving. Then on a cone of* z, $Lp^\Sigma(z) = Lp^{\Gamma,\Sigma}(z)$.

Proof. Fix $(\mathcal{P}, \Sigma) \in \Gamma$ and suppose on a cone of z, there is $\mathcal{M}_z \lhd Lp^\Sigma(z)$ such that letting Φ_z be the iteration strategy of $\mathcal{M}_z$ (as a Σ-mouse), $\Phi_z \notin \Gamma$.

Claim 5.5 *y is in a Σ-mouse over $(\mathcal{P}, x)$. Furthermore, whenever $(\mathcal{Q}, \Sigma_\mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, y is in a $\Sigma_\mathcal{Q}$-mouse over $(\mathcal{Q}, x)$.*

Proof. Let Γ^* be a good, scaled pointclass such that Δ_{Γ^*} contains $\wp(\mathbb{R}) \cap L_{\alpha+1}(\Gamma, \mathbb{R})$ and the function $z \mapsto \Phi_z$. Let $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures a universal Γ^* set (and other necessary sets like Σ etc) for some $z \geq_T x, y$. Let $(\mathcal{N}_\alpha : \alpha \leq \delta_z)$ be the models in the $L[E, \Sigma][\mathcal{P}, x]$-construction of $\mathcal{N}^*_z$ and $\mathcal{N} = \mathcal{N}_{\delta z}$.

Let $(\delta_i : i < \omega)$ enumerate the first ω Woodin cardinals of $\mathcal{N}$ and let $\lambda = sup_i\delta_i$. Let M be the derived model at λ as computed in $\mathcal{N}^*_z[g]$ for some generic $g \subseteq Col(\omega, <\lambda)$. Let $w \in \mathcal{N}^*_z[g]$ code $\mathcal{N}|\delta_0, g \restriction \delta_0$ and $\mathcal{N}|\delta_0, g \restriction \delta_0$ codes w. There is some $\mathcal{Q} \lhd \mathcal{N}$ such that $\mathcal{Q}[w]$ is equivalent to $\mathcal{M}_w$ and that $\Phi_w \in M$ and therefore (the interpretation of) Γ is in M. So $M \vDash y \in OD(x)$. This implies $y \in \mathcal{N}$ as desired.

The furthermore clause is similar. □

Using the claim, we let for any $(\mathcal{Q}, \Sigma_\mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\mathcal{M}_\mathcal{Q}$ be the least $\Sigma_\mathcal{Q}$-mouse over $(\mathcal{Q}, x)$ containing y and $\Phi_\mathcal{Q}$ be its strategy. Let A$\subseteq \mathbb{R}$ code the set $\{(\mathcal{M}_\mathcal{Q}, \Phi_\mathcal{Q}) : (\mathcal{Q}, \Sigma_\mathcal{Q}) \in I(\mathcal{P}, \Sigma)\}$.

Let Γ^* be a good pointclass such that $\Sigma, x \mapsto Lp(x), A \in \Delta_{\Gamma^*}$ and let $u \in \mathbb{R}$ be such that $(\mathcal{N}^*_u, \delta_u, \Sigma_u)$ Suslin captures a universal Γ^* set. Let $\mathcal{N}$ be the last model of the $L[E][x]$-construction in $\mathcal{N}^*_u|\delta_u$. By Theorem 1.3, there is $(\mathcal{Q}, \Sigma_\mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $\Sigma_\mathcal{Q} \restriction \mathcal{N} \in L[\mathcal{N}]$. By universality of $\mathcal{N}$, noting we can compare $\mathcal{M}_\mathcal{Q}$ vs Nin $\mathcal{N}^*_u$, $\mathcal{M}_\mathcal{Q} \lhd \mathcal{N}$. Therefore, $y \in \mathcal{N}$. This contradicts our assumption that y does not belong to any mouse over x.

□

Now we finish the proof of Lemma 1.4. Let A be the set of hod pairs or sts hod pairs $(\mathcal{P}, \Sigma)$ such that $\mathsf{Code}(\Sigma) \in \Gamma$ and Σ has strong branch condensation and is strongly fullness preserving.

[29] $(\mathcal{P}, \Sigma)$ may be anomalous as defined in [5, 7]. Here "generate" means: if $(\mathcal{P}, \Sigma)$ is a hod pair, then $\Gamma(\mathcal{P}, \Sigma) = \Gamma$ and if $(\mathcal{P}, \Sigma)$ is an sts pair, then $\Gamma^b(\mathcal{P}, \Sigma) = \Gamma$.

Claim 5.6 *$A \neq \emptyset$. Furthermore, if $(\mathcal{P}, \Sigma) \in A$, then there is a hod pair $(Q, \Lambda) \in A$ such that λ^Q is a successor ordinal and $(Q^-, \Lambda_{Q^-}) \in I(\mathcal{P}, \Sigma)$.*

Proof. To see $A \neq \emptyset$. Let Γ' be a good pointclass such that $Mice \in \Delta_{\Gamma'}$ and there is sjs $\vec{C} = (C_i : i < \omega) \in \Delta_{\Gamma'}$ such that $C_0 = Mice$. Note that by Lemma 5.4, $Mice \in \Gamma$. Let z be such that $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Mice$ and $\vec{C}$ as in Theorem 2.7. Then the first model $(\mathcal{P}_0, \Sigma_0)$ of the hod pair construction of $\mathcal{N}_z^*$ exists. Let $(P, \Sigma) = (\mathcal{P}_0, \Sigma_0)$. We have that Σ is fullness preserving and has branch condensation. Moreover, $Code(\Sigma) \in \Gamma^*$ as otherwise $\Gamma \subseteq L(\Sigma, \mathbb{R})$. Hence, $(\mathcal{P}, \Sigma) \in A$.

Now suppose $(\mathcal{P}, \Sigma) \in A$. There is a β such that the hod pair construction of $\mathcal{N}_z^*$ (possibly a different coarse Woodin mouse from the above) reaches a pair $(\mathcal{P}_\beta, \Sigma_\beta) \in I(\mathcal{P}, \Sigma)$. The pair (Q, Λ) is the next hod pair $(\mathcal{P}_{\beta+1}, \Sigma_{\beta+1})$ in this construction. Such a pair exists provided

1. $\mathcal{N}_{\beta+1}$ doesn't project across $\delta^{\mathcal{P}_\beta} =_{def} \delta_\beta$,
2. if $\beta = \gamma + 1$ then $\mathcal{N}_{\beta+1} \vDash \delta_\beta$ is Woodin,
3. if β is limit, no levels of $\mathcal{N}_{\beta+1}$ projects across δ_β and $(\delta_\beta^+)^{\mathcal{N}_\beta} = (\delta_\beta^+)^{\mathcal{P}_\beta}$.

We can rule out each case by standard arguments (e.g. see the argument in the proof of [5, Theorem 6.1]). For example, if in (1), there is a level $Q \lhd \mathcal{N}_{\beta+1}$, that projects across δ_β, letting Λ be its strategy. The key point is (Q, Λ) is an anomalous hod pair; so using the proof of Lemma 4.4, we can show $L(\Lambda, \mathbb{R}) \vDash \mathsf{MC}(\Lambda) + \forall \mathcal{R} \lhd^{\mathsf{c}}_{\mathsf{hod}} \mathcal{P}\ \mathsf{MC}(\Lambda_{\mathcal{R}})$. This allows us to analyze HOD of $L(\Lambda, \mathbb{R})$ and show that letting $A \subseteq \rho_1(\mathcal{R}) < \delta_\beta$[30] be Σ_1-definable over $\mathcal{R}$ from $p_1(\mathcal{R})$ and $A \notin \mathcal{R}$, then $A \in OD(\Lambda_{\mathcal{R}(\beta)} = \Sigma_\beta$. By $\mathsf{MC}(\Sigma)$ in $L(\Lambda, \mathbb{R})$ and the fact that $\mathcal{P}_\beta \lhd \mathcal{R}$ is full, $A \in \mathcal{R}$. Contradiction. □

It follows from the above claim that if

$$\Gamma_1 = \bigcup_{(\mathcal{P},\Sigma)\in A} \Gamma(\mathcal{P}, \Sigma)$$

then

(1) Γ_1 is a mouse full pointclass such that for some limit ordinal α there is a sequence of mouse full pointclasses $(\Gamma_\beta : \beta < \alpha)$ such that for $< \gamma < \alpha$, $\Gamma_\beta \lhd_{mouse} \Gamma_\gamma$ and $\Gamma_1 = \bigcup_{\beta<\alpha} \Gamma_\beta$.

It follows from Theorem 5.3 that there is a possibly anomalous hod pair $(\mathcal{P}, \Sigma)$ such that either

1. $\mathcal{P}$ is of lsa type and $\Gamma^b(\mathcal{P}, \Sigma) = \Gamma_1$ or
2. $\mathcal{P}$ is not of lsa type and $\Gamma(\mathcal{P}, \Sigma) = \Gamma_1$.

Because $\Gamma \vDash \mathsf{SMC}$ and because $\Gamma_1 \trianglelefteq_{mouse} \wp(\mathbb{R})$, we must have that Σ is strongly fullness preserving (for instance see [4, Lemma 6.21]). Notice that we get a hod pair as opposed to an sts pair. This is because we have good pointclasses beyond Γ.

Notice also that $\mathsf{Code}(\Sigma) \notin \Gamma$, as otherwise it follows from the claim that $(\mathcal{P}, \Sigma) \in A$. Thus, it must be the case that $\mathcal{P}$ is an anomalous hod premouse. We now get a standard contradiction as in the proof of Theorem 6.1 of [4], where it is argued that the computation of $HOD^{L(\Sigma,\mathbb{R})}$ gives a contradiction.

[30]For simplicity, assume $\rho_1(\mathcal{R}) < \delta_\beta$.

6 Proof of Theorem 1.3

We outline the main ideas in the proof of Theorem 1.3.

6.1 Basic notions and main ideas

We are in fact working towards the proof of Theorem 1.3, and the notation and the terminology of this subsection will be used in the later subsections. Fix $(\mathcal{P}, \Sigma)$, Γ_1, F and z as in the statement of Theorem 1.3. Let $\mathcal{N} = (\mathsf{Le}(\emptyset))^{\mathcal{N}_z^*}$. ⊣

Goal: We are looking for $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.

We start working in $\mathcal{N}_z^*$. Without loss of generality we can assume that

(1) whenever $\mathcal{R} \in pB(\mathcal{P}, \Sigma) \cap (\mathcal{N}_z^* | \underset{\cdot}{z})$ there is $\mathcal{S} \in pI(\mathcal{R}, \Sigma_{\mathcal{R}}) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{S}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.

As in [4], there are several cases.

1. $\mathcal{P}$ is of successor type.
2. $\mathcal{P}$ is of limit type and $\mathcal{P}$ is meek.
3. $\mathcal{P}$ is non-meek but $\mathcal{P}$ is not of #-lsa type.
4. $(\mathcal{P}, \Sigma)$ is an sts hod pair.

The first two cases are just like the cases considered in [4, Theorem 6.5], i.e. the "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular" case. For the remaining two cases we need more ideas to be discussed below.

Definition 6.1 *Suppose for a moment that we are working in some model of* ZFC. *Suppose κ is an inaccessible cardinal. We say that $(\mathcal{Q}, \Lambda)$ is a* ***hod pair at*** *κ if*

1. $(\mathcal{Q}, \Lambda)$ *is a hod pair,*
2. $\mathcal{Q} \in \mathsf{HC}$[31]
3. Λ *is a (κ, κ)-iteration strategy,*
4. $\mathsf{Code}(\Lambda)$ *is a κ-universally Baire set of reals.*

⊣

Suppose $(\mathcal{Q}, \Lambda)$ is a hod pair at κ. Then we let

$Lp^{\Lambda,\kappa}(a) = \bigcup\{\mathcal{M} : \mathcal{M}$ is a sound Λ-mouse over a such that $\rho_\omega(\mathcal{M}) = \mathsf{ord}(a)$ and $\mathcal{M} \trianglelefteq (\mathsf{Le}((\mathcal{Q}, \Lambda), a)^{V_\kappa}\}$.

As is customary, we let $\mathsf{Lp}_\alpha^{\Lambda,\kappa}(a)$ be the αth iterate of $\mathsf{Lp}^{\Lambda,\kappa}(a)$. Below $\mathcal{S}^*(\mathcal{R})$ is the $*$-transform of $\mathcal{S}$ into a hybrid mouse over $\mathcal{R}$, it is defined when $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ (cf. [8]).

Definition 6.2 (Fullness preservation in models of ZFC) *Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair at κ. We then say Σ is κ-fullness preserving if the following holds for all $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma) \cap V_\kappa$.*

[31] We will later apply this definition to $\mathcal{Q}$ which are not countable. The reason we make this assumption is so that we can have clause 4 below. It follows that the current definition makes sense in a variety of situations, and in particular when clause 4 holds after collapsing $\mathcal{Q}$ to be countable.

1. *For all meek layers $\mathcal{R}$ of Q such that $\mathcal{R}$ is of successor type*[32], *letting $\mathcal{S} = \mathcal{R}^{-}$*[33], *for all $\eta \in (\mathrm{ord}(\mathcal{S}), \mathrm{ord}(\mathcal{R}))$ if η is a cutpoint cardinal of $\mathcal{R}$ then*

$$(\mathcal{R}|(\eta^+)^{\mathcal{R}})^* = \mathsf{Lp}^{\Sigma_{\mathcal{S},\mathcal{T}},\kappa}(\mathcal{R}|\delta).$$

2. *For all meek layers $\mathcal{R}$ of Q such that $\mathcal{R}$ is of limit type,*

$$\mathcal{R} = \mathsf{Lp}^{\Sigma_{\mathcal{R}|\delta^{\mathcal{R}},\mathcal{T}},\kappa}(\mathcal{R}|\delta^{\mathcal{R}}).$$

3. *If $\mathcal{P}$ is of #-lsa type then $\mathsf{Lp}^{\Sigma^{stc}_{Q,\mathcal{T}},\kappa}(Q) \vDash$ "δ^Q is a Woodin cardinal"*[34].

We continuing our work inside some model of ZFC.

Definition 6.3 (Universal tail) *Suppose (Q, Λ) is a hod pair at κ such that Λ has branch condensation and is κ-fullness preserving. Suppose $\lambda < \kappa$ is an inaccessible cardinal. Then we say (Q^*, Λ^*) is a **λ-universal tail** of (Q, Λ) if there is a (possibly generalized) stack*

$$\mathcal{T} = (\mathcal{M}_\beta, \mathcal{T}_\beta, E_\beta : \beta < \lambda)$$

on Q according to Λ with last model Q^* such that $\mathrm{lh}(\mathcal{T}) = \lambda$ and for any $(\mathcal{S}, \mathcal{R}) \in I(Q, \Lambda) \cap V_\lambda$ there is a stack $\mathcal{U}$ on $\mathcal{R}$ according to $\Lambda_{\mathcal{R},\mathcal{S}}$ such that for some $\alpha <$, $\mathcal{M}_\alpha$ is the last model of $\mathcal{U}$.

*If $\mathcal{T}$ is as above then we say $\mathcal{T}$ is a **λ-universal stack** on Q according to Λ.*

Now observe that because of our assumption on $(\mathcal{P}, \Sigma)$, whenever $Q, \mathcal{R} \in pI(\mathcal{P}, \Sigma)$, (Q, Σ_Q) and $(\mathcal{R}, \Sigma_{\mathcal{R}})$ have a common tail in $\mathcal{N}^*_z|\delta_z$. In fact more is true. Suppose κ is a strong cardinal of $\mathcal{N}^*_z$. Then it follows that if $Q, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}^*_z|\kappa$ then (Q, Σ_Q) and $(\mathcal{R}, \Sigma_{\mathcal{R}})$ have a common tail in $\mathcal{N}^*_z|\kappa$. This means that whenever $\kappa < \delta_z$ is a cardinal of $\mathcal{N}^*_z$ and $Q \in (pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)) \cap \mathcal{N}^*_z|\kappa$, we can form the direct limit of all Σ_Q iterates of Q that are in $\mathcal{N}^*_z|\kappa$. Let $\mathcal{R}^{Q,\Sigma_Q}_\kappa$ be this direct limit.

Lemma 6.4 (Uniqueness of universal tails) *Suppose $Q \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}^*_z|\delta_z$. Then for each $\mathcal{S} \lhd^c_{hod} Q$ and $\mathcal{N}$-strong $\kappa < \delta_z$ such that $\mathcal{S} \in \mathcal{N}^*_z|\kappa$, there is a unique κ-universal tail of $(\mathcal{S}, \Sigma_{\mathcal{S}})$. In fact, letting $\mathcal{R} = \mathcal{R}^{\mathcal{S},\Sigma_{\mathcal{S}}}_\kappa$, $(\mathcal{R}, \Sigma_{\mathcal{R}})$ is the unique κ-universal tail of $(\mathcal{S}, \Sigma_{\mathcal{S}})$*

Definition 6.5 *Suppose $Q \in (pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)) \cap \mathcal{N}^*_z|\delta_z$ and κ is an $\mathcal{N}$-strong cardinal such that $Q \in \mathcal{N}^*_z|\kappa$. Then we say $\mathcal{N}$ **captures** a tail of (Q, Σ_Q) below κ if there is a hod pair $(\mathcal{R}, \Lambda) \in \mathcal{N}$ such that Λ is a (κ, κ)-iteration strategy and there is a term relation $\tau \in \mathcal{N}^{Coll(\omega,<\kappa)}$ such that whenever $g \subseteq Coll(\omega, |\mathcal{R}|^+)$ is $\mathcal{N}$-generic,*

1. *$\mathcal{N}[g] \vDash$ "$(\mathcal{R}, \tau_g)$ is a hod pair at κ such that τ_g is κ-fullness preserving" and $\tau_g \restriction \mathcal{N} = \Lambda$,*

2. *for some $\lambda < \kappa$, $\mathcal{R} = \mathcal{R}^{Q,\Lambda}_\lambda$ and letting $T, U \in \mathcal{N}[g]$ witness that τ_g is κ-uB, whenever $h \subseteq Coll(\omega, <\kappa)$ is $\mathcal{N}[g]$-generic, $(p[T])^{\mathcal{N}[g][h]} = \mathsf{Code}(\Sigma_{\mathcal{R}}) \cap \mathcal{N}[g][h]$.*

We say $\mathcal{N}$ captures $B(Q, \Sigma_Q)$ below κ if whenever $\mathcal{R} \in pB(Q, \Sigma_Q) \cap \mathcal{N}^*_z|\kappa$, $\mathcal{N}$ captures $(\mathcal{R}, \Sigma_{\mathcal{R}})$ below κ.

Towards a contradiction, we assume that $\mathcal{N}$ does not capture a tail of $(\mathcal{P}, \Sigma)$

[32] See Definition **??**.

[33] This is the longest proper layer of $\mathcal{R}$.

[34] Here, if Σ is a short tree strategy then $\Sigma^{stc} = \Sigma$.

Notation 6.6 *For each $Q \in pB(\mathcal{P}, \Sigma)$, we let λ_Q be the least $\mathcal{N}$-strong cardinal ν such that $\mathcal{N}$ captures the ν-universal tail of (Q, Σ_Q). We let $(\mathcal{R}^{Q,\Sigma}, \Phi^{Q,\Sigma})$ be the λ_Q-universal tail of (Q, Σ_Q). For each inaccessible cardinal ν such that $Q \in \mathcal{N}|\nu$, we let $(\mathcal{R}^{Q,\Sigma}_\nu, \Phi^{Q,\Sigma}_\nu)$ be the ν-universal tail of (Q, Σ_Q). If $\lambda \geq \lambda_Q$ then $\pi^{\Sigma_Q}_{Q,\mathcal{R}^{\mathcal{P},\Sigma}_\lambda}$ is the iteration map $\pi^{\Sigma_Q}_{Q,\mathcal{R}^{Q,\Sigma}}$.*

Notation 6.7 *Suppose now that κ_0 is an $\mathcal{N}$-strong cardinal that reflects the set of $\mathcal{N}$-strong cardinals. Let*

$$\mathcal{E}_0 = \{E \in \vec{E}^{\mathcal{N}} : \mathcal{N} \vDash \text{“}\nu(E) \text{ is inaccessible" and for all } \eta \in (0, \nu(E)),\ \mathcal{N} \vDash \text{“}\eta \text{ is a strong cardinal" if and only if } Ult(\mathcal{N}, E) \vDash \text{“}\eta \text{ is a strong cardinal"}\}.$$

⊣

Notation 6.8 *Working in $\mathcal{N}$, let*

$$\mathcal{F} = \{(Q, \Lambda) : Q \in \mathcal{N}|z \wedge \mathcal{J}[\mathcal{N}] \vDash \text{“}(Q, \Lambda) \text{ is a hod pair at } \delta_z \text{ and } \Lambda \text{ has branch condensation and is } \delta_z\text{-fullness preserving”}\}.$$

We have that $\mathcal{F}$ is a directed system. Let for $\lambda \leq \delta_z$,

$$\mathcal{F} \restriction \lambda = \{(Q, \Lambda) \in \mathcal{F} : Q \in \mathcal{N}|\lambda\}.$$

We let $\mathcal{R}^*$ be the direct limit of $\mathcal{F} \restriction \kappa_0$ under the iteration maps. ⊣

Definition 6.9 *Let $\mathcal{R}_0 = (\mathcal{R}^{\mathcal{P},\Sigma}_{\kappa_0})^b$.* ⊣

The next lemma summarizes what was proved in [4].

Lemma 6.10 *The following holds.*

1. *Suppose $Q \in pB(\mathcal{P}, \Sigma) \cap \mathcal{N}^*_z|\kappa_0$. Then $\mathcal{R}^{Q,\Sigma_Q} \in \mathcal{N}|\kappa_0$.*
2. *Suppose $Q \in pB(\mathcal{P}, \Sigma)$, $\lambda > \kappa_0$ is a strong cardinal of $\mathcal{N}$ such that $Q \in \mathcal{N}|\lambda$, and $E \in \mathcal{E}_0$ is an extender with critical point κ_0 such that $\nu(E) > (\lambda^+)^{\mathcal{N}^*_z}$. Then $\Phi^{Q,\Sigma} \restriction Ult(\mathcal{N}, E) \in Ult(\mathcal{N}, E)$.*
3. *Let $\mathcal{R}^*$ be as in Notation 6.8. Then either $\mathcal{R}_0 \trianglelefteq_{hod} \mathcal{R}^*$ or $\mathcal{R}_0|\delta^{\mathcal{R}_0} = \mathcal{R}^*$. Moreover, $\mathcal{R}_0 \in \mathcal{N}$ and $\Sigma_{\mathcal{R}_0} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.*

The idea of the proof of Lemma 6.10 is explained in the next section.

6.2 The meek case

In the case $\lambda^{\mathcal{P}}$ is a successor ordinal. We assume Σ is strongly guided by $\vec{B} = \{B_i : i < \omega\}$ and some tail $(\mathcal{R}, \Phi)$ of $(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ is captured by $\mathcal{N}$. We can then capture an iterate of $\mathcal{P}$ by using the universality of the $L[E, \Phi]^{\mathcal{N}}$-construction. In particular, let κ be the least $< \delta$-strong of $\mathcal{N}$ above $\mathcal{R}$, and $(\mathcal{M}, \Psi) = (\mathcal{R}^{\mathcal{P}}_\kappa, \Psi^{\mathcal{P}}_\kappa)$. One can show $\mathcal{M} \in \mathcal{N}$ by showing $\mathcal{M}|\delta^{\mathcal{M}}$ is V^{HOD}_Θ as computed in the derived model of $\mathcal{N}_1$ at κ, where $\mathcal{N}_1 = L[\mathcal{N}^*_1]$ and $\mathcal{N}^*_1$ is the last model of the $L[E, \Phi]^{\mathcal{N}}$-construction. To see $\Psi \restriction \mathcal{N} \in \mathcal{N}$, $\mathcal{N}$ can define the following strategy Λ and verify that $\Lambda = \Psi \restriction \mathcal{N}$.

Let $Q = \mathcal{R}^{\mathcal{P}}_\kappa$. Given normal tree $\mathcal{T} \in dom(\Lambda)$, we let $\Lambda(\mathcal{T}) = b$ if one of the following holds:

1. $\mathcal{T}$ is based on Q^- and $b = \Psi_{Q^{\mathcal{T}}}$.[35]

[35] $Q^- = Q(\lambda^Q - 1)$ if λ^Q is a successor ordinal.

2. $\mathcal{T}$ has is not entirely based on Q^- and if $(\mathcal{S}, \mathcal{U})$ are such that $\mathcal{T}$ up to $\mathcal{S}$ is based on Q^- and $\mathcal{U}$ is on $\mathcal{S}$ above $\mathcal{S}^-$, then one of the following holds:

   (a) $\mathcal{U}$ has a fatal drop at (α, γ) and letting $\mathcal{W}$ be the part of $\mathcal{U}$ after stage α, $\mathcal{W}^\frown\{\mathcal{M}_b^{\mathcal{U}}\}$ is according to the strategy of $O_\gamma^{\mathcal{M}_\alpha^{\mathcal{U}}}$.

   (b) $\mathcal{U}$ doesn't have a fatal drop, $Q(\mathcal{U}, b)$ exists and $Q(\mathcal{U}, b)$ is an initial segment of $L[E, \Psi_{\mathcal{S}^-}]^{\mathcal{N}}$.

   (c) None of the above holds. There is an extender $E \in \vec{E}^{\mathcal{N}}$ with $crit(E) = \kappa$ and such that $\mathcal{T} \in \mathcal{N}|\nu(E)$ and there is $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \pi_E(Q)$ with the property that $\pi_E \restriction Q = \sigma \circ \pi_b^{\mathcal{T}}$.

Suppose $\lambda^{\mathcal{P}}$ is a limit ordinal and without loss of generality, we assume $cof^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$. Recall we assume the theorem holds for every $Q \in pB(\mathcal{P}, \Sigma)$. The following facts take place in $\mathcal{N}$ and are easy to prove. The key point in all of these proofs is that letting E^* be the resurrection extender of E, then $\pi_{E^*}(\Sigma \restriction \mathcal{N}_z^*) = \Sigma \restriction Ult(\mathcal{N}_z^*, E^*)$.

Lemma 6.11 *Suppose ν is an inaccessible cardinal and $\lambda > \nu$ is strong in $\mathcal{N}$. Suppose $(Q, \Sigma_Q) \in B(\mathcal{P}, \Sigma) \cap \mathcal{N}_z^*|\nu$, $\mathcal{R}_\nu^Q \in \mathcal{N}$, and $\Psi_\nu^Q \restriction \mathcal{N}|\lambda \in \mathcal{N}$. Then $\Psi_\nu^Q \restriction \mathcal{N}|\delta \in \mathcal{N}$.*

Lemma 6.12 *Suppose $(Q, \Sigma_Q) \in B(\mathcal{P}, \Sigma) \cap \mathcal{N}_z^*|\kappa_0$. Then $\lambda_Q < \kappa_0$ and therefore $\mathcal{R}^Q \in \mathcal{N}|\kappa$.*

Lemma 6.13 *Suppose $(Q, \Sigma_Q) \in B(\mathcal{P}, \Sigma) \cap \mathcal{N}_z^*$ and $\lambda > \kappa$ is a strong cardinal such that $\lambda_Q < \lambda$. Let $E \in \mathcal{E}$ be such that $crt(E) = \kappa$ and $\nu(E) > (\lambda^+)^{\mathcal{N}_z^*}$. Then $\Psi^Q \restriction Ult(\mathcal{N}, E)|\delta \in Ult(\mathcal{N}, E)$.*

The above lemmas easily imply parts 1 and 2 of Lemma 6.10. To see part 3, first note that $\mathcal{R}_0 = (\mathcal{R}_{\kappa_0}^{\mathcal{P},\Sigma}$. The first clause of 3 is clear from our hypothesis. To see $\mathcal{R}_0 \in \mathcal{N}$, let $\Lambda = \oplus_{\alpha<\lambda^{\mathcal{R}_0}} \Sigma_{\mathcal{R}_0}(\alpha)$. We claim that $\Lambda \restriction \mathcal{N}|\delta \in \mathcal{N}$. This implies $\mathcal{R}_0 \in \mathcal{N}$ because $\mathcal{R}_0 = Lp_\omega^\Lambda(\mathcal{R}_0|\delta^{\mathcal{R}_0})$ and $Lp_\omega^\Lambda(\mathcal{R}_0|\delta^{\mathcal{R}_0}) \in \mathcal{N}$ by universality.

To see that $\Lambda \in \mathcal{N}$, note that the sequence $(\Sigma_{\mathcal{R}_0(\alpha)} \restriction \mathcal{N}|\delta : \alpha < \lambda^{\mathcal{R}_0}\} \in \mathcal{N}$ because for each such α, we can let $(Q, \Phi) \in \mathcal{F}_0$ such that $(\mathcal{R}_0(\alpha), \Sigma_{\mathcal{R}_0(\alpha))}$ is an iterate of (Q, Φ). $\mathcal{N}$ can define a κ_0-universal stack $\vec{\mathcal{S}} \in \mathcal{N}$ that witnesses this. Then $\Sigma_{\mathcal{R}_0(\alpha)} = \Phi_{\vec{\mathcal{S}}, \mathcal{R}_0(\alpha)}$ and doesn't depend on the choice of (Q, Φ).

We have outlined the argument that $\oplus_{\alpha<\lambda^{\mathcal{R}_0}} \Sigma_{\mathcal{R}_0}(\alpha) \restriction \mathcal{N}|\delta \in \mathcal{N}$. Now we need to argue $\Sigma_{\mathcal{R}} \restriction \mathcal{N}|\delta \in \mathcal{N}$. We will define a π_E-realizable strategy Λ of $\mathcal{R}$ in $\mathcal{N}$ and show that $\Lambda = \Sigma_{\mathcal{R}} \restriction \mathcal{N}|\delta$.

We briefly review definitions and notations related to the analysis of stacks in [4, Section 6.2]; see [4, Section 6.2] for a more detailed discussion. Suppose $\mathcal{P}$ is a hod premouse and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. Let $\mathcal{S}$ be a model that appears in $\vec{\mathcal{T}}$. By $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ we mean the part of $\vec{\mathcal{T}}$ up to and including $\mathcal{S}$ (according to the tree order of $\vec{\mathcal{T}}$), we define $\vec{\mathcal{T}}_{\geq \mathcal{S}}, \vec{\mathcal{T}}_{<\mathcal{S}}, \vec{\mathcal{T}}_{>\mathcal{S}}$ similarly. We let $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ be the normal components of $\vec{\mathcal{T}}$, i.e. $\mathcal{M}_0 = \mathcal{P}$, $\mathcal{T}_\alpha$ is a normal tree on $\mathcal{M}_\alpha$, and $\mathcal{M}_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}$. We say $\mathcal{R}$ is a *terminal node* of $\vec{\mathcal{T}}$ if for some α,β, $\mathcal{R} = \mathcal{M}_\beta^{\mathcal{T}_\alpha}$ and $\pi_{0,\beta}^{\mathcal{T}_\alpha}$ is defined. We say $\mathcal{R}$ is a *non-trivial terminal node* of $\vec{\mathcal{T}}$ if letting (α,β) witness that $\mathcal{R}$ is a terminal node of $\vec{\mathcal{T}}$, the extender $E_\beta^{\mathcal{T}_\alpha}$ is applied to $\mathcal{R}$ in the tree $\mathcal{T}_\alpha$ to obtain the model $\mathcal{M}_{\beta+1}^{\mathcal{T}_\alpha}$. We write $tn(\vec{\mathcal{T}})$ for the set of terminal nodes of $\vec{\mathcal{T}}$ and $ntn(\vec{\mathcal{T}})$ for the set of non-trivial terminal nodes of $\vec{\mathcal{T}}$.

For $Q, \mathcal{R} \in tn(\vec{\mathcal{T}})$, we write $Q \prec^{\vec{\mathcal{T}}} \mathcal{R}$ if the Q-to-$\mathcal{R}$ iteration embedding in $\vec{\mathcal{T}}$ exists, and we write $\pi_{Q,\mathcal{R}}^{\vec{\mathcal{T}}}$ for this embedding. We write $Q \prec^{\vec{\mathcal{T}},s} \mathcal{R}$ if letting $\vec{\mathcal{U}}$ be the part of $\vec{\mathcal{T}}$ between Q and $\mathcal{R}$, then $\vec{\mathcal{U}}$ is an iteration on Q. We write $\vec{\mathcal{T}}_{Q,\mathcal{R}}$ for $\vec{\mathcal{U}}$.

Let $C \subseteq tn(\vec{\mathcal{T}})$. We say C is *linear* (*strongly linear* respectively) if C is linearly ordered by $\prec^{\vec{\mathcal{T}}}$ ($\prec^{\vec{\mathcal{T}},s}$ respectively). We say C is *closed* if C is strongly linear and whenever α is a limit point of C, then letting $\mathcal{R}$ be the direct limit of $C \restriction \alpha$ (under the iteration embeddings), we have $\mathcal{R} \in C$. We say C is *cofinal* if for every $\mathcal{S} \in \vec{\mathcal{T}}$, there are $Q, \mathcal{R} \in C$ such that $Q \prec^{\vec{\mathcal{T}},s} \mathcal{R}$ and $\mathcal{S}$ is in $\vec{\mathcal{T}}_{Q,\mathcal{R}}$. Note that if $\vec{\mathcal{T}}$ doesn't have a last model, but there is a strongly closed and cofinal $C \subseteq tn(\vec{\mathcal{T}})$, then C uniquely determines a cofinal branch of $\vec{\mathcal{T}}$. If such a C doesn't exist, then η is a successor ordinal, say $\eta = \alpha + 1$. Let $\mathcal{U} = \vec{\mathcal{T}}_\alpha$ and $D = \{\mathcal{S} \in tn(\mathcal{U}) : \mathcal{U}_{\geq \mathcal{S}}$ is a tree on $\mathcal{S}\}$. In this case D has a $\prec^{\vec{\mathcal{T}},s}$-largest element and we write $\mathcal{S}_{\vec{\mathcal{T}}}$ for this element. Then $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is a normal tree based on $\mathcal{S}_{\vec{\mathcal{T}}}(\beta+1)$ and above $\delta_\beta^{\mathcal{S}_{\vec{\mathcal{T}}}}$ for some $\beta < \lambda^{\mathcal{S}_{\vec{\mathcal{T}}}}$. We write $\xi^{\vec{\mathcal{T}},\mathcal{S}_{\vec{\mathcal{T}}}}$ for $\delta_\beta^{\mathcal{S}_{\vec{\mathcal{T}}}}$ and similar notations are applied to any $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$.

Definition 6.14 (**π_E-realizable iterations**) *Let $\vec{\mathcal{T}}$ be a stack on $\mathcal{R}$. We say $\vec{\mathcal{T}}$ is **π_E-realizable** for $E \in \mathcal{E}_0$ if there is a strong cardinal $\lambda < \nu(E)$ such that $\vec{T} \in \mathcal{N}|\lambda$ and sequences $\langle \sigma_\mathcal{Q} : \mathcal{Q} \in tn(\vec{\mathcal{T}})\rangle$, $((\mathcal{S}_\mathcal{Q}, \Lambda_\mathcal{Q}) \in \mathcal{F}_0 \restriction \lambda : \mathcal{Q} \in tn(\vec{\mathcal{T}}))$ such that*

1. *$\sigma_\mathcal{R} = \pi_E \restriction \mathcal{R}$; for all $\mathcal{Q} \in tn(\vec{\mathcal{T}})$, $\sigma_\mathcal{Q} : \mathcal{Q} \to \pi_E(\mathcal{R})$.*
2. *For $\mathcal{Q}, \mathcal{S} \in tn(\vec{\mathcal{T}})$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{S}$, $\sigma_\mathcal{Q} = \sigma_\mathcal{S} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{S}}$.*
3. *For every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$, $\sigma_\mathcal{Q}[\mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1)] \subset rng(\pi^{\Lambda_\mathcal{Q}}_{\mathcal{S}_\mathcal{Q},\infty})$. We let $\mathcal{S}^*_\mathcal{Q} = \sigma_\mathcal{Q}(\psi^{\vec{\mathcal{T}},\mathcal{Q}}+1)$.*
4. *For every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$, letting $(\mathcal{S}_\mathcal{Q}, \Lambda_\mathcal{Q})$ be as above, and letting $k_\mathcal{Q} : \mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1) \to \mathcal{S}_\mathcal{Q}$ be given by: $k_\mathcal{Q}(x) = y$ if and only if $\sigma_\mathcal{Q}(x) = \pi^{\Lambda_\mathcal{Q}}_{\mathcal{S}_\mathcal{Q},\infty}(y)$ and $k_\mathcal{Q}\vec{\mathcal{T}}_\mathcal{Q}$ is according to $\Lambda_\mathcal{Q}$. Furthermore, $(\mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1), \Lambda^{k_\mathcal{Q}}_\mathcal{Q}) \in \pi_E(\mathcal{F}_0)$ and $\sigma_\mathcal{Q} \restriction \mathcal{Q}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1)$ is the embedding given by $\Lambda^{k_\mathcal{Q}}_\mathcal{Q}$.*
5. *$\mathcal{Q}, \mathcal{S} \in ntn(\vec{\mathcal{T}})$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{S}$,*
$$(\Lambda^{k_\mathcal{S}}_\mathcal{S})_{\mathcal{S}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{S}}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1)} = (\Lambda^{k_\mathcal{Q}}_\mathcal{Q})_{\mathcal{S}(\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{S}}(\xi^{\vec{\mathcal{T}},\mathcal{Q}}+1)}.$$
6. *For every trivial terminal node $\mathcal{Q}$, for every $\xi < \lambda^\mathcal{Q}$, there is a hod pair $(\mathcal{S}_\mathcal{Q}, \Lambda_\mathcal{Q}) \in \mathcal{F}_0 \restriction \lambda$ such that $\sigma_\mathcal{Q}(\xi+1) \subset rng(\pi^{\Lambda_\mathcal{Q}}_{\mathcal{S}_\mathcal{Q},\infty})$.*

Definition 6.15 *Let $\vec{\mathcal{T}} \in \mathcal{N}|\delta$ be a stack of on $\mathcal{R}$.*[36] *We let $\vec{\mathcal{T}} \in dom(\Lambda)$ iff for some $\xi < \delta$, whenever $E \in \mathcal{E}_0$ such that $\nu(E) > \xi$, $\vec{\mathcal{T}}$ is π_E-realizable . Define $\Lambda(\vec{\mathcal{T}}) = b$ iff for some $\xi < \delta$, whenever $E \in \mathcal{E}_0$ such that $\nu(E) > \xi$, $\vec{\mathcal{T}}^\frown b$ is π_E-realizable.*

The following are the key ideas in showing $\Lambda = \Sigma_\mathcal{R} \restriction \mathcal{N}|\delta$.

Claim 6.16 *Let $\vec{\mathcal{T}} \in dom(\Lambda)$. There is $\xi < \delta$ such that for all $E \in \mathcal{E}_0$ with $\nu(E) > \xi$*[37]*, letting $E^* \in E^{\mathcal{N}^*_z}$ be E's resurrection, and $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor map, for every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$, $i \circ \sigma_\mathcal{Q}$ is the iteration embedding according to $\Sigma_\mathcal{Q}$.*

Proof. We sketch the proof here. $\pi_{E^*} = i \circ \sigma_\mathcal{R}$ is the iteration embedding according to $\Sigma_\mathcal{R}$. Suppose $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$ and there is a largest $\mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$ (the case there is no largest $\mathcal{Q}^* \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$ is easy). Note that $\vec{\mathcal{T}}_{\mathcal{Q}^*,\mathcal{Q}}$ is based on $\mathcal{Q}^*({}^{\vec{\mathcal{T}},\mathcal{Q}^*}+1)$ and is according to $\Sigma_{\mathcal{Q}^*}$. Write ζ for ${}^{\vec{\mathcal{T}},\mathcal{Q}^*}$. Since

$$\sigma_\mathcal{Q}(x) = \sigma_{\mathcal{Q}^*}(f)(\sigma_\mathcal{Q}(a))$$

for $x = \pi^{\vec{\mathcal{T}}}_{\mathcal{Q}^*,\mathcal{Q}}(f)(a)$ and $f \in \mathcal{Q}^*$ and $a \in \mathcal{Q}(\zeta+1)$.

It is enough to show $\sigma_\mathcal{Q} \restriction \mathcal{Q}(\zeta+1)$ is according to $\Sigma_\mathcal{Q}$. For this, it suffices (by the inductive hypothesis, using item 5) to see that

$$(\Lambda^{k_{\mathcal{Q}^*}}_{\mathcal{Q}^*})_{\mathcal{Q}^*(\zeta+1)} = \Sigma_{\mathcal{Q}^*(\zeta+1)} \restriction \mathcal{N}|\delta. \tag{1}$$

The following are the main things to note:

[36] $\vec{\mathcal{T}}$ either has a strongly linear, closed and cofinal set $C \subseteq tn(\vec{\mathcal{T}})$ or $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is of limit length.

[37] We can take ξ to be above the sup of $\lambda_{\mathcal{Q}(\xi)}$ for every $\xi < \lambda_\mathcal{Q}$ and every $\mathcal{Q} \in ntn(\vec{\mathcal{T}})$.

(i) $Ult(\mathcal{N}, E) \vDash (\Lambda^{k_{Q^*}}_{Q^*})_{Q^*(\zeta+1)} = (\Lambda_{Q^*})^{\sigma_{Q^*}}_{\mathcal{S}^*_{Q^*}(\sigma_{Q^*}(\zeta+1))} \restriction \mathcal{N}|\nu$ where $\nu < \nu(E)$ is a strong cardinal such that $Q^* \in \mathcal{N}|\nu$.

(ii) $i((\Lambda_{Q^*})_{\mathcal{S}^*_{Q^*}(\sigma_{Q^*}(\zeta+1))}) = \Sigma_{\mathcal{S}^*_{Q^*}(\sigma_{Q^*}(\zeta+1))} \restriction \mathcal{N}|\delta$.

(ii) follows from the choice of E that allows us to find a common iterate of $(\mathcal{S}^*_{Q^*}, \Lambda_{Q^*})$ and $(\mathcal{R}^{Q^*}, \Psi^{Q^*} \restriction Ult(\mathcal{N}, E))$ in $\mathcal{N}|\lambda$. Since $i \circ \sigma_{Q^*}$ is the iteration embedding according to Σ_{Q^*}, $i((\Lambda_{Q^*})_{\mathcal{S}^*_{Q^*}(\sigma_{Q^*}(\zeta+1))}) = \Sigma_{Q^*(\zeta+1)} \restriction \pi_{E^*}(\mathcal{N})$, and $i \restriction \lambda = id$, (1) easily follows. □

Claim 6.17 *Suppose $\vec{\mathcal{T}}$ is π_E-realizable for E as in Claim 6.16. Then $\vec{\mathcal{T}} \in dom(\Sigma_{\mathcal{R}})$.*

Proof. We need to show that for any $Q \in tn(\vec{\mathcal{T}})$, $\vec{\mathcal{T}}_{\mathcal{R},Q}$ is according to $\Sigma_{\mathcal{R}}$ and if $\mathcal{S}_{\vec{\mathcal{T}}}$ exists then $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is according to $\Sigma_{\mathcal{S}_{\vec{\mathcal{T}}}}$.

The first claim is proved as follows. Let E^* be the resurrection of E and $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor map. Then

$$\pi_{E^*} \restriction \mathcal{R} = i \circ \sigma_Q \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{R},Q}.$$

Since $\pi_{E^*} \restriction \mathcal{R}$ is the iteration embedding according to $\Sigma_{\mathcal{R}}$, $\vec{\mathcal{T}}_{\mathcal{R},Q}$ is according to $\Sigma_{\mathcal{R}}$ by branch condensation.

The second claim has been proved by the previous claim. Note that letting $\mathcal{K} = \mathcal{S}_{\vec{\mathcal{T}}}$, $\vec{\mathcal{T}}_{\mathcal{K}}$ is based on $\mathcal{K}(\xi^{\vec{\mathcal{T}},\mathcal{K}} + 1)$ and is according to $\Lambda^{k_{\mathcal{K}}}_{\mathcal{K}}$. But the claim above shows that $\Lambda^{k_{\mathcal{K}}}_{\mathcal{K}} = \Sigma_{\mathcal{K}(\xi^{\vec{\mathcal{T}},\mathcal{K}}+1)}$. This is what we want. □

Claim 6.18 *Suppose $\vec{\mathcal{T}} \in dom(\Lambda)$, then $\Lambda(\vec{\mathcal{T}})$ is defined.*

Proof. Let ξ witness $\vec{\mathcal{T}} \in dom(\Lambda)$. First suppose $\mathcal{S}_{\vec{\mathcal{T}}}$ is undefined. So there is a closed and unbounded $C \subseteq ntn(\vec{\mathcal{T}})$. Let $b = b_C$ be the cofinal branch given by C. Let ξ^* be the sup of $\Lambda_{\mathcal{K}(\xi)}$ for $K \in ntn(\vec{\mathcal{T}}^\frown b$ and $\xi < \lambda^{\mathcal{K}}$. If $\lambda > max(\xi, \xi^*)$, then λ witnesses $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b \in dom(\Lambda)$.

Suppose $\mathcal{S}_{\vec{\mathcal{T}}}$ exists. Let $Q = \mathcal{S}_{\vec{\mathcal{T}}}$ and $\mathcal{T} = \vec{\mathcal{T}}_Q$ and $b = \Lambda_Q(k_Q\mathcal{T})$. It is easy to see (using arguments similar to the above claims) that b is independent of E and $\vec{\mathcal{T}}^\frown b$ is according to $\Sigma_{\mathcal{R}}$. Suppose π_b exists and let ξ^* be at least the sup of $\Lambda_{\mathcal{K}(\xi)}$ for $K \in ntn(\vec{\mathcal{T}}^\frown b$. Let $E\mathcal{E}_0$ be such that $\nu(E) > \lambda$ for some strong cardinal $\lambda > \xi^*$. Let $\mathcal{S} = \mathcal{M}^{\mathcal{T}}_b$ and $\Sigma_{\mathcal{S}} : \mathcal{S} \to \pi_E(\mathcal{R})$ be the natural map: $\sigma_{\mathcal{S}}(x) = \sigma_Q(f)(\tau(a))$ for $f \in Q$, $a \in (\delta(\mathcal{T}))^{<\omega}$ such that $x = \pi_b(f)(a)$, and $\tau(a) = \pi^{\Lambda^{k_Q}_Q}_{\mathcal{S}(\pi^{\vec{\mathcal{T}}}_b(\xi^{\vec{\mathcal{T}}}+1),\mathcal{S}_Q}(a)$. We want to verify that all clauses of Definition 6.14 hold. All are easy except possibly clause 6. Let E^* be E's resurrection extender and $i : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor map. Letting $j = i \circ \sigma_{\mathcal{S}}$, then j is according to $\Sigma_{\mathcal{S}}$. For every $\gamma < \lambda^{\mathcal{S}}$,

$$j[\mathcal{S}(\gamma)] \subseteq \pi^{\Psi^{\mathcal{S}(\gamma)}}_{\mathcal{S}(\gamma),\pi_{E^*}(\mathcal{S}^*_Q)(\sigma_{\mathcal{S}}(\gamma))}.$$

This means $(\mathcal{R}^{\mathcal{S}(\gamma)}, \Psi^{\mathcal{S}(\gamma)} : \gamma < \lambda^{\mathcal{S}})$ witnesses clause 6.

□

6.3 The non-meek case

Below we will develop a technology for recovering the full iterate of $\mathcal{P}$. Let $\mathcal{R}_0^+ = \mathcal{R}_{\kappa_0}^{\mathcal{P},\Sigma}$ be the iterate of $\mathcal{P}$ extending $\mathcal{R}_0$ and let $i : \mathcal{P} \to \mathcal{R}_0^+$ be the iteration embedding. We will recover an iterate of $\mathcal{R}_0^+$ inside $\mathcal{N}$ as an output of a backgrounded construction that is done over $\mathcal{R}_0$. Such constructions are called mixed hod pair constructions. The details of this construction appear in Section **??**.

There are two kinds of extenders that we will use in this construction. The extenders with critical point $> \delta^{\mathcal{R}_0}$ will have traditional background certificates. We will use the total extenders on the sequence of $\mathcal{N}$ to certify such extenders. The extenders with critical point $\delta^{\mathcal{R}_0}$ will come from a different source. The following key lemma illustrates the idea.

Lemma 6.19 *Let $\delta = \delta^{\mathcal{R}_0}$. Suppose $\mathcal{S} \in pI(\mathcal{R}_0^+, \Sigma_{\mathcal{R}_0^+})$ is a normal iterate of $\mathcal{R}_0^+$ that is obtained by iterating entirely above $\delta^{\mathcal{R}_0}$. Suppose that $\alpha \in \mathrm{dom}(\vec{E}^{\mathcal{S}})$ is such that letting $E =_{def} \vec{E}^{\mathcal{S}}(\alpha)$, $\mathrm{crit}(E) = \delta$, $\mathcal{S}|\alpha \in \mathcal{N}$ and $\Sigma_{\mathcal{S}|\alpha} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$. Then $E \in \mathcal{N}$. Moreover, $(a, A) \in E$ if and only if $a \in \nu_E^{<\omega}$, $A \in [\delta]^{|a|}$ and whenever $F \in \mathcal{E}_0$ is such that $\mathrm{crit}(F) = \kappa_0$ and*

$$\mathcal{N} \models \text{“there is a strong cardinal } \lambda \text{ in the interval } (\kappa_0, \nu_F) \text{ such that } \mathcal{S} \in \mathcal{N}|\lambda\text{”},$$

$\pi^{\Sigma_{\mathcal{S}|\alpha}}_{\mathcal{S}|\alpha, \pi_F(\mathcal{R}_0)}(a) \in \pi_F(A)$[38].

Proof. Set $\mathcal{M}^+ = Ult(\mathcal{R}_0^+, E)$ and $\mathcal{M} = Ult(\mathcal{R}_0, E)$. Let F^* be the resurrection of F and let $\sigma : Ult(\mathcal{N}, F) \to \pi_{F^*}(\mathcal{N})$ be the canonical factor map. We have that $\sigma \restriction \nu_F = id$. Thus, $\pi_{F^*} \restriction \mathcal{N} = \sigma \circ \pi_F$. It follows that $\pi_{F^*} \restriction \mathcal{R}^+$ is the iteration embedding implying

(1) $\pi_{F^*} \restriction \mathcal{R}_0^+ = \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}_0^+)} \circ \pi_E^{\mathcal{R}_0^+}$.

We now have that

$$\begin{aligned}
(a, A) \in E \quad &\iff \quad a \in \pi_E^{\mathcal{R}_0^+}(A) \\
&\iff \quad \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}_0^+)}(a) \in \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}_0^+)}(\pi_E^{\mathcal{R}_0^+}(A)) \\
&\iff \quad \sigma(\pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R}_0)}(a)) \in \pi_{F^*}(A) \\
&\iff \quad \sigma(\pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R}_0)}(a)) \in \sigma(\pi_F(A)) \\
&\iff \quad \pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R}_0)}(a) \in \pi_F(A)
\end{aligned}$$

Therefore,

$$(a, A) \in E \iff \pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R}_0)}(a) \in \pi_F(A).$$

By our assumption, the right hand side of the equivalence can be computed in $\mathcal{N}$. Hence $E \in \mathcal{N}$. □

Thus, the extenders with critical point $\mathcal{R}_0$ that we will use in our mixed hod pair construction have the following property. If Q is the current level of the construction and Λ is its strategy then let E be the set of pairs (a, A) such that $a \in (\mathcal{R}_0)^{<\omega}$ and for every $F \in \mathcal{E}_0$ such that $\mathrm{crit}(F) = 0$ and

$$\mathcal{N} \models \text{“there is a strong cardinal ł in the interval } (0, \nu_F) \text{ such that } Q \in \mathcal{N}|\text{ł”},$$

[38]The embedding $\pi^{\Sigma_{\mathcal{S}|\alpha}}_{\mathcal{S}|\alpha, \pi_F(\mathcal{R}_0)}$ is just $\pi^{\Sigma_{\mathcal{S}|\alpha}}_{\mathcal{S}|\alpha, \mathcal{R}^{\mathcal{S}|\alpha, \Sigma_{\mathcal{S}_\alpha}}_{\pi_F(0)}}$. We will often abuse our notation this way.

$\pi^{\Lambda}_{Q,\pi_F(\mathcal{R}_0)}(a) \in \pi_F(A)$.

There is one problem with this approach. We need to know the strategy Λ of Q before we can find the relevant extender. To resolve this problem, we will first define the strategy Λ. Essentially Λ will pick branches that, for some η, are π_E-realizable for all $E \in \mathcal{E}_0$ such that $\nu_E > \eta$. We will call such strategies $\mathcal{E}_0$-certified.

Lemma 6.20 *Suppose $\eta >¸0$ is such that $\mathcal{N} \vDash$ "η is a strong cardinal that reflects the set of strong cardinals". Set $\mathcal{S}^+ = \mathcal{R}^{\mathcal{P},\Sigma}_{\eta}$, $i^+ = \pi^{\Sigma_{\mathcal{R}_0^+}}_{\mathcal{R}_0^+,\mathcal{S}^+}$, $\mathcal{S} = (\mathcal{S}^+)^b$ and $i = i^+ \restriction \mathcal{R}_0$. Then $i \in \mathcal{N}$ and $\mathcal{N} \vDash |\mathcal{S}| < (\eta^+)^{\mathcal{N}}$.*

Remark 6.21 *We also get, by methods in [7, Theorem 9.2.2], that $\pi_E \restriction \mathcal{S}$ is a strongly condensing set in $Ult(\mathcal{N}, E)[g]$ where $g \subseteq Coll(\omega, \pi_E(\eta))$ is any $Ult(\mathcal{N}^*_z, E)$-generic.*

Proof. If suffices show that $i \in \mathcal{N}$. Let $F \in \vec{E}^{\mathcal{N}}$ be any extender such that $\text{crit}(F) = \kappa_0$ and $Ult(\mathcal{N}, F) \vDash$ "η is a strong cardinal". Let F^* be the background certificate of F and let $k : Ult(\mathcal{N}, F) \to \pi_{F^*}(\mathcal{N})$ be the canonical factor map. We now have that $\pi_{F^*} \restriction \mathcal{R}_0^+ = \pi^{\Sigma_{\mathcal{R}_0^+}}_{\mathcal{R}^+,\pi_{F^*}(\mathcal{R}_0^+)}$. We thus have that

(1) $\pi_{F^*} \restriction \mathcal{R}_0^+ = \pi^{\Sigma_{\mathcal{S}^+}}_{\mathcal{S}^+,\pi_{F^*}(\mathcal{R}_0^+)} \circ \pi^{\Sigma_{\mathcal{R}_0^+}}_{\mathcal{R}_0^+,\mathcal{S}^+}$.

Let $m = \pi^{\Sigma_{\mathcal{S}^+}}_{\mathcal{S}^+,\pi_{F^*}(\mathcal{R}_0^+)} \restriction \mathcal{S}|\delta^{\mathcal{S}}$. We have that

(2) $m = \pi^{\Sigma_{\mathcal{S}|\delta^{\mathcal{S}}}}_{\mathcal{S}|\delta^{\mathcal{S}},\pi_{F^*}(\mathcal{R}_0)|\xi}$ where $\xi = \sup(m[\delta^{\mathcal{S}}])$.

Because $\Sigma_{\mathcal{S}|\delta^{\mathcal{S}}} \restriction \mathcal{N} \in \mathcal{N}$, we have that $k(\Sigma_{\mathcal{S}|\delta^{\mathcal{S}}} \restriction \mathcal{N}) = \Sigma_{\mathcal{S}|\delta^{\mathcal{S}}} \restriction \pi_{F^*}(\mathcal{N})$ and therefore, $m \in \pi_{F^*}(\mathcal{N})$ and $m \in \text{rge}(k)$. Let $n = k^{-1}(m)$. Thus,

(3) $n = \pi^{\Sigma_{\mathcal{S}|\delta^{\mathcal{S}}}}_{\mathcal{S}|\delta^{\mathcal{S}},\pi_F(\mathcal{R}_0)|k^{-1}(\xi)}$.

Notice now that for $x \in \mathcal{R}_0$,

(4) $\pi_{F^*}(x) = \pi^{\Sigma_{\mathcal{S}^+}}_{\mathcal{S}^+,\pi_{F^*}(\mathcal{R}_0^+)}(i(x))$

implying that

(5) $\mathcal{S}$ is the transitive collapse of $\{\pi_{F^*}(f)(m(a)) : f \in \mathcal{R}_0 \wedge a \in (\delta^{\mathcal{S}})^{<\omega}\}$ and $\pi^{\Sigma_{\mathcal{S}^+}}_{\mathcal{S}^+,\pi_{F^*}(\mathcal{R}_0^+)} \restriction \mathcal{S}$ is the inverse of the transitive collapse.

(5) now implies that

(6) $\mathcal{S}$ is the transitive collapse of $\{\pi_F(f)(n(a)) : f \in \mathcal{R}_0 \wedge a \in (\delta^{\mathcal{S}})^{<\omega}\}$.

Since $\{\pi_F(f)(n(a)) : f \in \mathcal{R}_0 \wedge a \in (\delta^{\mathcal{S}})^{<\omega}\} \in \mathcal{N}$, we have that if $\sigma : \mathcal{S} \to \pi_F(\mathcal{R}_0)$ is the inverse of the transitive collapse then $\sigma \in \mathcal{N}$. Moreover,

(7) $\pi_{F^*} \restriction \mathcal{R}_0 = k \circ \sigma \circ i$ and $k \circ \sigma = \pi^{\Sigma_{\mathcal{S}^+}}_{\mathcal{S}^+,\pi_{F^*}(\mathcal{R}_0^+)} \restriction \mathcal{S}$.

It now follows that

(8) $i(x) = \sigma^{-1}(\pi_F(x))$.

Since both σ and π_F are in $\mathcal{N}$, we get that $i \in \mathcal{N}$.

$\square$

Suppose now that κ is an $\mathcal{N}$-strong cardinal that reflects the set of $\mathcal{N}$-strong cardinals such that $\kappa > \kappa_0$. Let

$$\mathcal{E} = \{E \in \vec{E}^{\mathcal{N}} : \mathcal{N} \vDash \text{“}\nu(E) \text{ is inaccessible" and for all } \eta \in (\kappa, \nu(E)), \mathcal{N} \vDash \text{“}\eta \text{ is a strong cardinal" if and only if } Ult(\mathcal{N}, E) \vDash \text{“}\eta \text{ is a strong cardinal"}\}.$$

Set $\mathcal{R}^+ = \mathcal{R}_\kappa^{\mathcal{P},\Sigma}$ and let $\mathcal{R} = (\mathcal{R}^+)^b$. It follows from Lemma 6.10 that $\mathcal{R} \in \mathcal{N}$. Let $\Phi^+ = (\Phi_\kappa^{\mathcal{P},\Sigma})_{\mathcal{R}|\delta^{\mathcal{R}}}$ and $\Phi = \Phi^+_{\mathcal{R}}$[39]

Notice that $\Phi \restriction \mathcal{N} \in L[\mathcal{N}]$.

Definition 6.22 *Working in $\mathcal{N}$, we say (σ, Q) is **$\mathcal{E}$-realizable** if*

- *$\sigma : \mathcal{R} \to Q$ is an elementary embedding,*
- *for some $\mathcal{N}$-strong cardinal $\xi \in (\kappa, z)$, $Q \in \mathcal{N}|\xi$ and for all $E \in \mathcal{E}$ such that $\xi < \nu(E)$ and for all $\mathcal{N}$-generic $g \subseteq Coll(\omega, Q)$, there is $j : Q \to \pi_E(\mathcal{R})$ such that $j \in Ult(\mathcal{N}, E)[g]$ and $\pi_E \restriction \mathcal{R} = \sigma \circ j$.*[40]

We say that j is (π_E, σ)-realizable. Continuing our work in $\mathcal{N}$, let $\mathcal{F}'_{\mathcal{E}}$ be the set of π_E-realizable pairs (σ, Q). Given $(\sigma, Q) \in \mathcal{F}'_{\mathcal{E}}$, let $\xi(\sigma, Q) < z$ witness that clause 2 above holds for (σ, Q). Given $E \in \mathcal{E}$ such that $\xi(\sigma, Q) < \nu(E)$, letting $j : Q \to \pi_E(\mathcal{R})$ be any (π_E, σ)-realizable embedding, set $\Psi_{\sigma,Q,E,j} = (j$-pullback of $\pi_E(\Phi))$[41]. ⊣

The following is an easy consequence of the remark after Lemma 6.20 and

Definition 6.23 *Working in $\mathcal{N}$, we say (σ, Q) is **neatly $\mathcal{E}$-realizable** if (σ, Q) is $\mathcal{E}$-realizable and for all $E_0, E_1 \in \mathcal{E}$ with $\nu(E_0) \leq \nu(E_1)$,*

$$\Psi_{\sigma,Q,E_0} \restriction \mathcal{N}|\nu(E_0) = \Psi_{\sigma,Q,E_1} \restriction \mathcal{N}|\nu(E_0).$$

Let $\mathcal{F}_{\mathcal{E}}$ be the set of neatly $\mathcal{E}$-realizable pairs, and for $(\sigma, Q) \in \mathcal{F}_{\mathcal{E}}$, let

$$\Psi_{\sigma,Q} = \cup\{\Psi_{\sigma,Q,E} : E \in \mathcal{E} \wedge \xi(\sigma, Q) < \nu(E)\}^{42}.$$

⊣

The following is a key lemma.

Lemma 6.24 *Suppose $\mathcal{S}$ is a Φ^+-iterate of $\mathcal{R}^+$ via $\mathcal{T}$ such that $\pi^{\mathcal{T},b}$ is defined and $\mathcal{S}^b \in \mathcal{N}$. Then $\pi^{\mathcal{T},b} \in \mathcal{N}$, $(\pi^{\mathcal{T},b}, \mathcal{S}^b)$ is neatly $\mathcal{E}$-realizable and*

$$\Psi_{\pi^{\mathcal{T},b},\mathcal{S}^b} = \Phi^+_{\mathcal{S}^b} \restriction \mathcal{N}.$$

[39] See Notation 6.6.

[40] Notice that $\pi_E \restriction \mathcal{R} \in Ult(\mathcal{N}, E)$, see Lemma 6.20.

[41] $\Psi_{\sigma,Q,E,j}$ is defined in $Ult(\mathcal{N}, E)$.

[42] $\Psi_{\sigma,Q}$ is defined in $\mathcal{J}[\mathcal{N}]$ and $\Psi_{\sigma,Q} \restriction \mathcal{N}$ is total.

Proof. The proof of $\pi^{\mathcal{T},b} \in \mathcal{N}$ is exactly the proof of Lemma 6.20. The proof of the fact that $(\pi^{\mathcal{T},b}, \mathcal{S}^b)$ is neatly $\mathcal{E}$-realizable is via a simple absoluteness argument. Let $E \in \mathcal{E}$ be such that $\mathcal{S}^b \in \mathcal{N}|\nu(E)$ and let E^* be the background certificate of E. Let $k : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the canonical factor map. We have that $\text{crit}(k) \geq \nu(E)$. Set $\sigma = \pi^{\mathcal{T},b}$. Notice that

(1) in $\pi_{E^*}(\mathcal{N})$, it is forced by $Coll(\omega, \mathcal{S}^b)$ that there is a (π_{E^*}, σ)-realizable $j : \mathcal{S}^b \to \pi_{E^*}(\mathcal{R})$, and
(2) if $g \subseteq Coll(\omega, \mathcal{S}^b)$ is $\pi_{E^*}(\mathcal{N})$-generic and $j : \mathcal{S}^b \to \pi_{E^*}(\mathcal{R})$ is any (π_{E^*}, σ)-realizable embedding, then the j-pullback of $\pi_{E^*}(\Phi)$ is $\Phi^+_{\mathcal{S}^b}$.

It follows that

(3) in $\mathcal{N}$, it is forced by $Coll(\omega, \mathcal{S}^b)$ that there is a (π_{E^*}, σ)-realizable $j : \mathcal{S}^b \to \pi_{E^*}(\mathcal{R})$, and
(4) if $g \subseteq Coll(\omega, \mathcal{S}^b)$ is $\mathcal{N}$-generic and $j : \mathcal{S}^b \to \pi_{E^*}(\mathcal{R})$ is any (π_{E^*}, σ)-realizable embedding, then the j-pullback of $\pi_E(\Phi)$ is independent of j.

Let Π in $Ult(\mathcal{N}, E)$ be the strategy of $\mathcal{S}^b$ such that it is forced by $Coll(\omega, \mathcal{S}^b)$, that for some (π_E, σ)-realizable j, Π is the j-pullback of $\pi_E(\Phi)$. Let $\tau : \mathcal{S}^b \to \pi_E(\mathcal{R})$ be defined by setting $\tau(x) = \pi_E(f)(\pi^{\Pi}_{\mathcal{S}|\mathcal{S}^b, \pi_E(\mathcal{R})}(a))$ where $x = \sigma(f)(a)$, $f \in \mathcal{R}$ and $a \in (\mathcal{S}^b)^{<\omega}$. It follows from clause 2 of Lemma 6.20 that τ is a (π_E, σ)-realization and $\tau \in Ult(\mathcal{N}, E)$. It then follows from (2) that $k(\Pi) = \Phi^+_{\mathcal{S}^b}$ and therefore, $\Pi = \Phi^+_{\mathcal{S}^b} \restriction Ult(\mathcal{N}, E)$. □

Definition 6.25 *Suppose $(\sigma, \mathcal{Q}) \in \mathcal{F}_{\mathcal{E}}$ and $E \in \mathcal{E}$ is such that $\xi(\sigma, \mathcal{Q}) < \nu(E)$. We say that $\tau = \tau^E_{\sigma,\mathcal{Q}}$ is the* ***canonical** E**-realization*** *of $(\sigma, \mathcal{Q})$ if $\tau : \mathcal{Q} \to \pi_E(\mathcal{R})$ and $\tau(x) = \pi_E(f)(\pi^{\Psi_{\sigma,\mathcal{Q},E}}_{\mathcal{Q}|\mathcal{Q},\mathcal{R}(\sigma,\mathcal{Q})}(a))$ where $\mathcal{R}(\sigma, \mathcal{Q}) \trianglelefteq_{hod} \mathcal{R}$ is the $\Psi_{\sigma,\mathcal{Q},E}$-iterate of $\mathcal{Q}|\mathcal{Q}$, $f \in \mathcal{R}$, $a \in (\mathcal{Q})^{<\omega}$ and $x = \sigma(f)(a)$.* ⊣

The following definitions use various terms from [7] that if introduced here, would make this note impossibly long. Please look them up in [7].

Definition 6.26 (π_E-realizable iterations) *Suppose*

1. *$\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$,*
2. *$\mathcal{T} \in \mathcal{N}$ is either a stack on $\mathcal{V}$ or an st-stack on $\mathcal{V}$*[43]*,*
3. *$E \in \mathcal{E}$.*

Suppose that

$$\mathcal{T} = ((\mathcal{M}_\alpha)_{\alpha<\eta}, (E_\alpha)_{\alpha<\eta-1}, D, R, (\beta_\alpha, m_\alpha)_{\alpha\in R}, T)$$

is a stack. Set $R^b = \{\alpha \in R : \pi^{\mathcal{T},b}_{0,\alpha}$ is defined$\}$. We say $\mathcal{T}$ is ***π_E-realizable*** *if the following holds:*

1. *$\mathcal{N} \vDash$ "λ is a strong cardinal".*
2. *$\mathcal{T} \in \mathcal{N}|\text{lh}(E)$.*
3. *For all $\alpha \in R^b$, $(\pi^{\mathcal{T}_{\leq\alpha},b}, \mathcal{M}^b_\alpha) \in \mathcal{F}_{\mathcal{E}}$*[44]*.*

[43] If $\mathcal{T}$ is an st-stack then $\mathcal{V}$ must be of #-lsa type.
[44] See Definition 6.23.

4. *For all $\alpha < \beta$ such that $\alpha, \beta \in R^b$, setting $\tau_\alpha = \tau^E_{\sigma,Q}$, $\tau_\alpha = \tau_\beta \circ \pi^{\mathcal{T},b}_{\alpha,}$*

.Forall$\alpha \in R^b$, letting $\Psi_\alpha = \Psi_{\sigma_\alpha, \mathcal{M}^b_\alpha}$,

5. (a) *if $\alpha \neq \max(R^b)$ and $\mathsf{nc}^{\mathcal{T}}_\alpha$ is based on $\mathcal{M}^b_\alpha | \mathcal{M}^b_\alpha$ then $\mathsf{nc}^{\mathcal{T}}_\alpha$ is according to Ψ_α,*

   (b) *if $\alpha = \max(R^b)$ and $\mathcal{U} = \downarrow (\mathcal{T}_{\geq\alpha}, \mathcal{M}^b_\alpha)$[45] then*

      i. *if $\mathcal{U}$ is based on $\mathcal{M}^b_\alpha$ and is above $\mathcal{M}^b_\alpha$ then it is according to the unique strategy Π of $\mathcal{M}^b_\alpha$ witnessing that $\mathcal{M}^b_\alpha$ is a Ψ_α-mouse over $\mathcal{M}^b_\alpha | \mathcal{M}^b_\alpha$, and*

      ii. *if $\mathcal{U}$ is based on $\mathcal{M}^b_\alpha | \mathcal{M}^b_\alpha$ then $\mathcal{U}$ is according to Ψ_α.*

*We say that $(\sigma_\alpha : \alpha \in R^b)$ are the π_E-**realizable embeddings** of $\mathcal{T}$ and $(\Psi_\alpha : \alpha \in R^b)$ are the π_E-**realizable strategies** of $\mathcal{T}$. We say $\mathcal{T}$ is $\mathcal{E}$-**realizable** if for some η, $\mathcal{T}$ is π_E-realizable for every $E \in \mathcal{E}$ with the property that $\mathrm{lh}(E) > \eta$.*

The definition of the above concepts for st-stacks is very similar. The embeddings σ_α are once again defined for $\alpha \in R^b$ which once again consists of those $\alpha < \mathrm{lh}(\mathcal{T})$ with the property that $\pi^{\mathcal{T},b}_{0,\alpha}$ is defined. We leave the details to the reader. ⊣

Recall the **$\mathcal{E}$-realizable backgrounded constructions** in [7, Definitionn 10.2.28]. We will use them to find the Q-structures of various iterations and help us define the $\mathcal{E}$-certified iterations.

Definition 6.27 *Suppose $\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$. Suppose $\mathcal{T} \in \mathcal{N}$ is a stack or an st-stack on $\mathcal{V}$ and $E \in \mathcal{E}$. We say $\mathcal{T}$ is E-**certified** if the following conditions are satisfied.*

1. *$\mathcal{T}$ is π_E-realizable.*

2. *Suppose $\tau \in (R^b)^{\mathcal{T}}$ is such that letting $\mathcal{U} =_{def} \mathsf{nc}^{\mathcal{T}}_\tau$, $\mathcal{U}$ is above $\mathcal{M}^b_\tau$. Let $\alpha < \mathrm{lh}(\mathcal{U})$ be a limit ordinal and let $c = [0, \alpha)_{\mathcal{U}}$. Then the following conditions hold.*

   (a) *If $\mathrm{m}^+(\mathcal{U} \restriction \alpha) \vDash$ "$\dot{(}\mathcal{U} \restriction \alpha)$ is not a Woodin cardinal"[46] then $Q(c, \mathcal{U} \restriction \alpha)$ exists and $Q(c, \mathcal{U} \restriction \alpha)\mathrm{m}^+(\mathcal{U} \restriction \alpha)$.*

   (b) *If $\mathrm{m}^+(\mathcal{U} \restriction \alpha) \vDash$ "$\dot{(}\mathcal{U} \restriction \alpha)$ is a Woodin cardinal" and there is $\mathcal{W}$ such that*

      i. *$\mathcal{W}$ appears on the $\mathsf{Le}^{\mathcal{E},c}(\mathrm{m}^+(\mathcal{U}| \restriction \alpha))$ construction of $\mathcal{N}$ and*

      ii. *$\mathcal{W} \vDash$ "$\dot{(}\mathcal{U} \restriction \alpha)$ is a Woodin cardinal" but $\mathcal{J}_\omega[\mathcal{W}] \vDash$ "$\dot{(}\mathcal{U} \restriction \alpha)$ is not a Woodin cardinal",*

      then $Q(c, \mathcal{U} \restriction \alpha)$ exists and $Q(c, \mathcal{U} \restriction \alpha) = \mathcal{W}$.

   (c) *The above two clauses fail. Then $\mathcal{T}$ is an st-stack, $\alpha + 1 = \mathrm{lh}(\mathcal{U})$ and $\tau + \alpha \in R^{\mathcal{T}} \cap \mathsf{max}^{\mathcal{T}}$.*

*We say that $\mathcal{T}$ is $\mathcal{E}$-**certified** if for some λ, $\mathcal{T}$ is E-certified for every $E \in \mathcal{E}$ such that $\mathrm{lh}(E) > \lambda$.* ⊣

And finally we define $\mathcal{E}$-certified strategies.

Definition 6.28 *Suppose $\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$. We let $\Lambda_{\mathcal{V}}$ be the partial strategy of $\mathcal{V}$ with the property that*

1. *$\mathrm{dom}(\Lambda_{\mathcal{V}})$ consists of $\mathcal{E}$-certified stacks $\mathcal{T}$ of limit length, and*

2. *for all $\mathcal{T} \in \mathrm{dom}(\Lambda_{\mathcal{V}})$, $\Lambda_{\mathcal{V}}(\mathcal{T}) = b$ if b is the unique x such that $\mathcal{T}^\frown\{x\}$ is $\mathcal{E}$-certified.*

We say $\Lambda_{\mathcal{V}}$ is the $\mathcal{E}$-certified strategy of $\mathcal{V}$. ⊣

[45] This is just the restriction of $\mathcal{T}_{\geq\alpha}$ to $\mathcal{M}^b_\alpha$.

[46] See Definition 3.1.

Suppose now that

$$\mathcal{T} = ((\mathcal{M}_\alpha)_{\alpha<\eta}, (E_\alpha)_{\alpha<\eta-1}, D, R, (\beta_\alpha, m_\alpha)_{\alpha\in R}, T)$$

is π_E-realizable as witnessed by $(\sigma_\alpha : \alpha \in R^b)$ and $(\Psi_\alpha : \alpha \in R^b)$. Using the language of [7, Chapter 9] applied in $Ult(\mathcal{N}, E)$ to $\pi_E(\mathcal{F} \upharpoonright \kappa)$, it is not hard to see that for $\alpha \in (R^b)^{\mathcal{T}}$, $\mathcal{M}^b_\alpha = \mathcal{P}_{Y_\alpha}$ where $Y_\alpha = \sigma_\alpha[\mathcal{M}^b_\alpha]$ and $\Psi_\alpha = \Sigma_{Y_\alpha}$. It now follows from the existence of condensing sets that there are unique $\mathcal{E}$-certified strategies.

Lemma 6.29 *Suppose $\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$. Suppose Λ and Ψ are two $\mathcal{E}$-certified strategies for $\mathcal{V}$. Then $\Lambda = \Psi$.*

Next, we need to show the correctness of the realizable strategy. The reader should review the notions of stacks for a strategy or a sts strategy in [7, Chapter 2].

Lemma 6.30 *Suppose $\mathcal{S}^*$ is a Φ^+-iterate of $\mathcal{R}^+$ via an iteration that is entirely above $\delta^{\mathcal{R}}$. Suppose further that $\mathcal{S} \lhd_{hod} \mathcal{S}^*$ is such that $\mathcal{S}^b = \mathcal{R}$ and $\mathcal{S} \in \mathcal{N}$. Let $\mathcal{T} \in \mathcal{N}$ be a stack on $\mathcal{S}$[47]. Suppose $\mathcal{T}$ is $\mathcal{E}$-certified. Then $\mathcal{T}$ is according to $\Phi^+_{\mathcal{S}}$. Thus, $\Lambda_{\mathcal{S}} = \Phi^+_{\mathcal{S}} \cap \mathcal{N}^2$.*[48]

Proof. Suppose

$$\mathcal{T} = ((\mathcal{M}_\alpha)_{\alpha<\eta}, (E_\alpha)_{\alpha<\eta-1}, D, R, (\beta_\alpha, m_\alpha)_{\alpha\in R}, T),$$

and suppose $\alpha \in R^b$ is such that $\mathcal{T}_{\leq\alpha}$ is according to $\Phi^+_{\mathcal{S}}$. We want to show that $\mathcal{U} = \mathsf{nc}^{\mathcal{T}}_\alpha$ is according to $\Phi^+_{\mathcal{M}_\alpha}$.

Suppose first that $\mathcal{U}$ is based on $\mathcal{M}^b_\alpha$[49]. Let $E \in \mathcal{E}$ be such that $\mathcal{T}$ is π_E-realizable as witnessed by $(\sigma_\alpha : \alpha \in R^b)$ and $(\Psi_\alpha : \alpha \in R^b)$. Let E^* be the background certificate of E and let $k : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the canonical factor map. Notice that for $\alpha \in R^b$,

(1) $k \upharpoonright \mathcal{N}|\xi = id$ where ξ is the least such that $\mathcal{T} \in \mathcal{N}|\xi$.
(2) In $\pi_{E^*}(\mathcal{N})$, $k(\sigma_\alpha) : \mathcal{M}^b_\alpha \to \pi_{E^*}(\mathcal{N})$ and $k(\Psi_\alpha)$ is the $k(\sigma_\alpha)$-pullback of $\pi_{E^*}(\Phi)$.
(3) $k(\sigma_\alpha) \upharpoonright \mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}$ is the iteration embedding according to $k(\Psi_\alpha)$.

Let F be the un-dropping extender of $\mathcal{T}_{\leq\alpha}$ and set $\mathcal{K}^+ = Ult(\mathcal{R}^+, F)$ and $j = \pi^{\Phi^+_{\mathcal{K}^+}}_{\mathcal{K}^+, \pi_{E^*}(\mathcal{R}^+)} \upharpoonright \mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}$. Notice now that

(4) $\Phi_{\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}}$ is the j-pullback of $\pi_{E^*}(\Phi)$ and j is the iteration embedding according to $\Phi_{\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}}$.

As the pairs $(k(\sigma_\alpha), k(\Psi_\alpha))$ and $(j, \Phi_{\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}})$ have the same property, it follows from Lemma **??** that $k(\sigma_\alpha) = j$ and $k(\Psi_\alpha) = \Phi_{\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}} \upharpoonright \pi_{E^*}(\mathcal{N})$. Since $k(\mathcal{U}) = \mathcal{U}$, in the case $\mathcal{U}$ is based on $\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}$, we have that $\mathcal{U}$ is according to $k(\Psi_\alpha)$ and therefore, $\mathcal{U}$ is according to $\Phi_{\mathcal{M}_\alpha|\mathcal{M}^b_\alpha}$, and in the case $\mathcal{U}$ is above $\delta^{\mathcal{M}^b_\alpha}$, we have that $\mathcal{U}$ is according to the unique strategy of $\mathcal{M}^b_\alpha$ that witnesses the fact that $\mathcal{M}^b_\alpha$ is a $\Phi_{\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}}$-mouse over $\mathcal{M}_\alpha|\delta^{\mathcal{M}^b_\alpha}$.

[47] We assume that $\mathcal{T}$ is a stack, but the proof works for generalized stacks as well.
[48] This equation does not imply that $\Lambda_{\mathcal{S}} = \Phi^+_{\mathcal{S}} \upharpoonright \mathcal{N}$, simply because it does not imply that if $x \in \mathrm{dom}(\Phi^+_{\mathcal{S}}) \cap \mathcal{N}$ then $\Phi^+_{\mathcal{S}}(x) \in \mathcal{N}$. To get the aforementioned equality, we need to show that $\Lambda_{\mathcal{S}}$ is total.
[49] There is yet another case: namely, $\alpha = \max R^b$ and $\mathcal{U} = \mathcal{T}_{\geq\alpha}$. But this case is very similar to our two cases.

Suppose now that $\mathcal{U}$ is above $\mathsf{ord}(\mathcal{M}^{\mathsf{b}}_\alpha)$. Here, we need to see that

(a) if $\beta < \mathrm{lh}(\mathcal{U})$ is a limit ordinal then letting $b = [0,\beta)_{\mathcal{U}}$, either $Q(b,\mathcal{U}) \lhd \mathrm{m}^+(\mathcal{U})$ or else $Q(b,\mathcal{U}) \lhd \mathsf{Lp}^{\Gamma,(\Phi^+)^{sts}_{\mathrm{m}^+(\mathcal{U})}}(\mathrm{m}^+(\mathcal{U}))$.

The following lemma establishes (a). For convenience, we will ignore the objects introduced above and treat next lemma in a general context. Thus $\mathcal{T}$ in the next lemma is not the $\mathcal{T}$ fixed above.

Lemma 6.31 *Suppose $\mathcal{T}$ is an $\mathcal{E}$-certified iteration of $\mathcal{S}$, $\alpha \in R^b$ and $\mathcal{U} = \mathsf{nc}^{\mathcal{T}}_\alpha$ is above $\mathsf{ord}(\mathcal{M}^{\mathsf{b}}_\alpha)$. Suppose further that $\beta < \mathrm{lh}(\mathcal{U})$ is a limit ordinal and $\mathcal{U}_{<\beta}$ is according to $\Phi^+_{\mathcal{M}_\alpha}$. Let $Q = \mathcal{M}^{\mathcal{U}}_\beta$ and $\eta > \delta^{Q^b}$ be such that $\mathcal{J}_\omega[(Q|\eta)^\#] \models$ "η is a Woodin cardinal" and let $\mathcal{W} \lhd Q$ be an sts mouse over $(Q|\eta)^\#$. Then $\mathcal{W}$ is a $(\Phi^+)^{stc}_{(Q|\eta)^\#}$-sts mouse.*

Proof. Towards a contradiction assume that $\mathcal{W}$ is not a $(\Phi^+)^{stc}_{(Q|\eta)^\#}$-sts mouse. It follows that $b = [0,\beta)_{\mathcal{U}}$ is not the branch chosen by $\Phi^+_{\mathcal{S}}$. For convenience, we change our notation and let $\mathcal{U}$ be $\mathcal{U} \restriction \beta$ and $Q = \mathrm{m}^+(\mathcal{U})$. It follows from Definition 6.27 that

(1) $\mathcal{W}$ is a model appearing in the fully backgrounded $\mathcal{E}$-realizable construction over $(Q|\eta)^\#$ done in $\mathcal{N}$.

What we need to see is that $\mathcal{W}$ is a $(\Phi^+)^{stc}_Q$-sts mouse over Q. To show this it is enough to show that every stack indexed in $\mathcal{W}$ is according to $(\Phi^+)^{stc}_Q$. To show this later fact, it is enough to show that

(b) if $t = (Q, \mathcal{U}_0, Q_1, \mathcal{U}_1)$ is an indexable stack[50] on Q appearing in the fully backgrounded $\mathcal{E}$-realizable construction over Q (done in $\mathcal{N}$) and c is the branch of t indexed in this construction then $t^\frown\{c\}$ is according to $(\Phi^+)^{stc}_Q$.

(b) is indeed enough. To see this, notice that if $s = (Q, \mathcal{U}'_0, Q'_1, \mathcal{U}'_1)$ is indexed in $\mathcal{W}$ and c' is the branch of s indexed in $\mathcal{W}$ then for some stack $t = (Q, \mathcal{U}_0, Q_1, \mathcal{U})$ as in (b) if e is the branch of t then $s^\frown\{c\}$ is a hull of $t^\frown\{e\}$. If t is according to $(\Phi^+)^{stc}_Q$ then it follows from hull condensation of $(\Phi^+)^{stc}_Q$ that s is also according to $(\Phi^+)^{stc}_Q$. We now work towards showing that t is according to $(\Phi^+)^{stc}_Q$.

Suppose first that $\mathcal{U}_0$ is according to $(\Phi^+)^{stc}_Q$. We then have that $\mathcal{U}_1$ is a stack based on Q^b_1. Because $(\mathcal{T}_{\leq\alpha})^\frown t$ is $\mathcal{E}$-certified, we can fix an extender $E \in \mathcal{E}$ such that $(\mathcal{T}_{\leq\alpha})^\frown t$ is π_E-realizable. We then have $\sigma : Q^b_1 \to \pi_E(\mathcal{R})$ such that $\pi_E \restriction \mathcal{R} = \sigma \circ \pi^{\mathcal{U}_0,b} \circ \pi^{\mathcal{T}_{\leq Q},b}$. We also have that $\mathcal{U}_1^\frown\{c\}$ is according to the σ-pullback of $\pi_E(\Phi_{\mathcal{R}})$. Therefore, t is according to $(\Phi^+)^{stc}_Q$.

It remains to show that $\mathcal{U}_0$ is according to $(\Phi^+)^{stc}_Q$. Without loss of generality, we assume that

- $\mathrm{lh}(\mathcal{U}_0) = \gamma + 1$,
- γ is a limit ordinal,
- $\mathcal{U}_0 \restriction \gamma$ is according to $(\Phi^+)^{stc}_Q$,
- $[0,\gamma)_{\mathcal{U}_0} \neq (\Phi^+)^{stc}_Q(\mathcal{U}_0)$,
- there is $\zeta \in R^{\mathcal{U}_0 \restriction \gamma}$ such that $(\mathcal{U}_0)_{\geq\zeta} = \mathsf{nc}^{\mathcal{U}_0}_\zeta$ and $\pi^{\mathcal{U}_0,b}$ exists,
- $\mathcal{J}_\omega[\mathrm{m}^+(\mathcal{U}_0)] \models$ "$\delta(\mathcal{U}_0)$ is a Woodin cardinal".

[50] See [7, Chapter3].

The last two clauses can be shown by examining the proof given for $\mathcal{U}_1$. Set $c_0 = [0,\gamma)_{\mathcal{U}_0}$, $\mathcal{Q}_0 =_{def} \mathcal{Q}$, $\mathcal{Q}_2 = \mathrm{m}^+(\mathcal{U}_0)$, $\mathcal{W}_0 =_{def} \mathcal{W}$ and $\mathcal{W}_2 = \mathcal{Q}(c_0, \mathcal{U}_0)$. We then have that

(2) $\mathcal{W}_2$ appears in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{Q}_2$ (done in $\mathcal{N}$).

Clearly (2) leads to an infinite descend. □

□

The following two lemmas show that when we do a sts hod pair constructions (in the sense of [7]), we don't terminate the construction because a branch indexed in the model fails to be according to $\Lambda_{\mathcal{V}}$. We prove the second lemma, whose proof relies on the fact that the $\mathcal{E}$-certified strategy of $\mathcal{V}$, is total. The proof of Lemma 6.32 is fairly similar to Lemma 6.34.

Lemma 6.32 *$\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$. Then $\Lambda_{\mathcal{V}}$ is total, and hence, $\Phi^+_{\mathcal{V}} \restriction \mathcal{N} = \Lambda_{\mathcal{V}}$.*

Lemma 6.33 *Suppose*

1. *$\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$,*
2. *$\mathcal{T} \in \mathcal{N}$ is either a stack on $\mathcal{V}$ or an st-stack on $\mathcal{V}$*[51]*,*
3. *$\pi^{\mathcal{T},b}$ is defined, $\mathcal{T}$ has a last model and $\mathcal{E}$-realizable.*

Let $\mathcal{S}$ be the last model of $\mathcal{T}$ and suppose $\mathcal{Q}$ is authenticated[52] *by $\mathcal{T}$ and is meek and of limit type*[53]*. Then $\mathcal{W}, \mathcal{U}, \sigma$ be as in [7, Definition 3.7.3] and letting $k : \mathcal{R} \to \mathcal{Q}$ be given by $k(x) = y$ if and only $\sigma^{-1}(\pi^{\mathcal{T},b}(x)) = \pi^{\mathcal{U}}(y)$, $(k, \mathcal{Q})$ is $\mathcal{E}$-realizable.*

Lemma 6.34 *Suppose*

1. *$\mathcal{V} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{V}^b$,*
2. *$\mathcal{T} \in \mathcal{N}$ is either a stack on $\mathcal{V}$ or an st-stack on $\mathcal{V}$*[54]*,*
3. *$\pi^{\mathcal{T},b}$ is defined, $\mathcal{T}$ has a last model and $\mathcal{E}$-realizable.*

Let $\mathcal{S}'$ be the last model of $\mathcal{T}$ and suppose $\eta < \mathsf{ord}(\mathcal{S}')$ is such that $\mathcal{J}_\omega[(\mathcal{S}'|\eta)^\#] \vDash$ "η is a Woodin cardinal". Suppose $\mathcal{S} =_{def} (\mathcal{S}'|\eta)^\#$ is such that $\mathcal{S}_{hod}\mathcal{S}'$ and $\mathcal{U} \in \mathcal{N}$ is an nuvs *stack according to $(\Lambda_{\mathcal{V}})_{\mathcal{S}}$ such that $\pi^{\mathcal{U},b}$ is defined. Let $\mathcal{Q} = \mathrm{m}^+(\mathcal{U})$ and suppose $t \in \mathcal{N}$ be an indexable stack on $\mathcal{Q}$ which is $(\mathcal{S}, (\Lambda_{\mathcal{V}})_{\mathcal{S}})$-authenticated*[55]*. Then $\mathcal{T}^\frown\mathcal{U}^\frown t$ is according to $\Lambda_{\mathcal{V}}$.*

Proof. Suppose $t = (\mathcal{Q}_0, \mathcal{X}_0, \mathcal{Q}_1, \mathcal{X}_1)$. Assume first that $\mathcal{X}_0$ is according to $(\Lambda_{\mathcal{V}})_{\mathcal{Q}}$. Set $p = \mathcal{T}^\frown\mathcal{U}^\frown\mathcal{X}_0$ and let $\sigma = \pi^{p,b}$. It follows from Lemma 6.30 that p is according to $\Lambda_{\mathcal{V}}$, and the previous lemma implies that $(\sigma, \mathcal{Q}_1^b) \in \mathcal{F}_{\mathcal{E}}$. Because $(\mathcal{Q}_1^b, \mathcal{X}_1)$ is a $(\mathcal{S}, (\Lambda_{\mathcal{V}})_{\mathcal{S}})$-authenticated iteration, it follows from Lemma 6.30 that $\mathcal{X}_1$ is according to $\Psi_{\sigma,\mathcal{Q}_1^b}$, and therefore, $\mathcal{T}^\frown\mathcal{U}^\frown t$ is according to $\Lambda_{\mathcal{V}}$.

[51]If $\mathcal{T}$ is an st-stack then $\mathcal{M}$ must be of lsa type.
[52]See [7, Definition 3.7.3].
[53]Thus, clause 3 of [7, Definition 3.7.3] holds.
[54]If $\mathcal{T}$ is an st-stack then $\mathcal{V}$ must be of #-lsa type.
[55]Notice that we, at this point, do not know that $\Lambda_{\mathcal{V}}$ is a total strategy in $\mathcal{J}[\mathcal{N}]$.

Thus, it is enough to show that $\mathcal{X}_0$ is according to $(\Lambda_{\mathcal{V}})_Q$. The argument given above implies that it is enough to show that for every $\alpha \in R^{\mathcal{X}_0}$ such that $\pi^{\mathcal{X}_0,b}_{0,\alpha}$ is defined and $\mathsf{nc}^{\mathcal{X}_0}_\alpha$ is a stack on $\mathcal{M}^{\mathcal{X}_0}_\alpha$ above $\mathsf{ord}((\mathcal{M}^{\mathcal{X}_0}_\alpha)^b)$ then $\mathsf{nc}^{\mathcal{X}_0}_\alpha$ is according to $(\Lambda_{\mathcal{V}})_{\mathcal{M}^{\mathcal{X}_0}_\alpha}$.

Assume then α is as above and $(\mathcal{X}_0)_{\leq\alpha}$ is according to $(\Lambda_{\mathcal{V}})_Q$. Set $\mathcal{M} = \mathcal{M}^{\mathcal{X}_0}_\alpha$, $\mathcal{X} = (\mathcal{X}_0)_{\leq\alpha}$ and let $\mathcal{Y} = \mathsf{nc}^{\mathcal{X}_0}_\alpha$. We want to see that $\mathcal{Y}$ is according to $(\Lambda_{\mathcal{V}})_{\mathcal{M}}$. Let $\beta < \mathrm{lh}(\mathcal{Y})$ be a limit ordinal such that $\mathcal{Y}_{<\beta}$ is according to $(\Lambda_{\mathcal{V}})_{\mathcal{M}}$. We want to see that if $b = [0,\beta)_{\mathcal{Y}}$ then $b = (\Lambda_{\mathcal{V}})_{\mathcal{M}}(\mathcal{Y}_{<\beta})$. The dificult case is when $Q(b, \mathcal{Y}_{<\beta})$ exists and is an sts mouse over $\mathrm{m}^+(\mathcal{Y}_{<\beta})$. In this case, we want to see that $Q(b, \mathcal{Y}_{<\beta})$ is a model appearing in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathrm{m}^+(\mathcal{U}_0)$ (done in $\mathcal{N}$). This would follows from the proof of the previous lemma. Our strategy for showing this is by showing (a) and (b) where these are the following statements:

(a) $Q(b, \mathcal{Y}_{<\beta})$ is a $(\Phi^+_{\mathrm{m}^+(\mathcal{Y}_{<\beta})})^{stc}$-mouse over $\mathrm{m}^+(\mathcal{Y}_{<\beta})$.

(b) If $\mathcal{W}$ is a $(\Phi^+_{\mathrm{m}^+(\mathcal{Y}_{<\beta})})^{stc}$-mouse over $\mathrm{m}^+(\mathcal{Y}_{<\beta})$ then $\mathcal{W}$ appears in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathrm{m}^+(\mathcal{U}_0)$ (done in $\mathcal{N}$). More precisely, letting

$$\mathsf{Le}^{\mathcal{E},c}(\mathrm{m}^+(\mathcal{Y}_{<\beta})) = (\mathcal{Z}_\gamma, \mathcal{K}_\gamma, F^+_\gamma, F_\gamma, b_\gamma : \gamma \leq \delta_z)$$

be the fully backgrounded $\mathcal{E}$-realizable construction over $\mathrm{m}^+(\mathcal{Y}_{<\beta})$ done in $\mathcal{N}$ then for some $\gamma < \delta_z$, $\mathcal{Z}_\gamma = \mathcal{W}$.

(a) is a consequence of strong branch condensation of Φ^+. (b) is a consequence of the fact that $\Lambda_{\mathcal{V}}$ is total, and hence $\Phi^+_{\mathcal{V}} \restriction \mathcal{J}[\mathcal{N}] = \Lambda_{\mathcal{V}}$ (see Lemma 6.30). Assuming that $\Lambda_{\mathcal{V}}$ is total, (b) can be proven by simply comparing $\mathcal{W}$ with the $\mathsf{Le}^{\mathcal{E},c}(\mathrm{m}^+(\mathcal{Y}_{<\beta}))$ construction. The stationarity of $\mathsf{Le}^{\mathcal{E},c}(\mathrm{m}^+(\mathcal{Y}_{<\beta}))$ implies that the construction side doesn't move, and the fact that $\Lambda_{\mathcal{V}}$ is total implies that the construction doesn't break down because in clause 3b of [7, Definitionn 10.2.28] we are unable to find the desired branch. The Important Anomaly stated in clause 3b of [7, Definitionn 10.2.28] does not occur (at least doesn't occur before reaching $\mathcal{W}$) because these type of branches are chosen internally and both the construction side and the $\mathcal{W}$-side must be choosing the same branch. But on the $\mathcal{W}$-side, the branch is according to $\Phi^+_{\mathcal{V}}$ and therefore, according to $\Lambda_{\mathcal{V}}$. In the next subsection, we will prove that $\Lambda_{\mathcal{V}}$ is total, and more details will be given. □

We devote this entire subsection to the definition of a construction producing the iterate of $\mathcal{R}^+$. In this construction, we use $\mathcal{E}$-certification method to acquire extenders with critical point $\delta^{\mathcal{R}}$, and we use the total extenders on the sequence of $\mathcal{N}$ to generate extenders with critical points $> \underset{\sim}{\mathcal{R}}$. First we define $\mathcal{E}$-certified extenders. The reader may wish to review Definition 6.25.

Definition 6.35 *Suppose $Q \in \mathcal{N}$ is a hod premouse such that Λ_Q (see Definition 6.28) is total and $Q^b = \mathcal{R}$. Suppose F is an extender such that $(Q, \tilde{F})$ is a reliable* lses *where $\tilde{F}$ is the amenable code of F. We say F is* ***$\mathcal{E}$-certified*** *if*

- *$(\pi_F \restriction \mathcal{R}, \pi_F(\mathcal{R})) \in \mathcal{F}_{\mathcal{E}}$ and*
- *for some $\mathcal{N}$-strong cardinal λ, for any $E \in \mathcal{E}$ such that* $\mathrm{lh}(E) > \lambda$*, setting $\tau = \tau_{\pi_F \restriction \mathcal{R}, \pi_F(\mathcal{R})}$*[56],

$$(a, A) \in F \iff \tau(a) \in \pi_E(A).$$

[56]See Definition 6.25.

*We say that τ is the E-**realizability map** of F.* ⊣

The next lemma shows that $\mathcal{E}$-certified extenders are on the sequence of $\mathcal{R}^+$ and its iterates.

Lemma 6.36 *Suppose $\mathcal{S}^* \in pI(\mathcal{R}^+, \Phi^+)$ and $\mathcal{S} \lhd_{hod} \mathcal{S}^*$ is such that $\mathcal{S} \in \mathcal{N}$ and $\mathcal{S}^b = \mathcal{R}$. Suppose F is such that $(\mathcal{S}, \tilde{F})$ is a reliable* lses [57] *where $\tilde{F}$ is the amenable code of F and F is $\mathcal{E}$-certified. Then $F \in \vec{E}^{\mathcal{S}^*}$.*

Proof. Let $\gamma = \mathsf{ord}(\mathcal{S})$ and suppose $F^* \in \vec{E}^{\mathcal{S}^*}(\gamma)$. Then F^* has exactly the same property as F and therefore, $F = F^*$. Thus, it is enough to show that $\gamma \in \mathrm{dom}(\vec{E}^{\mathcal{S}^*})$. Suppose first that there is $\gamma' \in \mathrm{dom}(\vec{E}^{\mathcal{S}^*})$ such that $\mathcal{S} \unlhd_{hod} \mathcal{S}^*|\gamma'$ and if $G' = \vec{E}^{\mathcal{S}^*}(\gamma')$ then $\mathrm{crit}(G') = \delta^{\mathcal{R}}$. Let γ^* be the least such γ' and set $G = \vec{E}^{\mathcal{S}^*}(\gamma^*)$. As F and G both have the property described in Definition 6.35, F is an initial segment of G, and therefore, $\gamma = \gamma^*$ and $\gamma \in \mathrm{dom}(\vec{E}^{\mathcal{S}^*})$. Suppose then that

(1) there is no $\gamma' \in \mathrm{dom}(\vec{E}^{\mathcal{S}^*})$ such that $\mathrm{crit}(\vec{E}^{\mathcal{S}^*}(\gamma')) = \delta^{\mathcal{R}}$.

Because F is $\mathcal{E}$-certified, we have that for some $\mathcal{N}$-strong cardinal ł, whenever $E \in \mathcal{E}$ is such that $\mathrm{lh}(E) > \lambda$, some proper initial segment of $\pi_E(\mathcal{R})$ is a $\Phi^+_{\mathcal{S}}$-iterate of $\mathcal{S}$. Therefore, $(\mathcal{S}, \Phi^+_{\mathcal{S}})$ is in HP^Γ, and hence, $(\mathcal{S}^*, \Phi^+_{\mathcal{S}^*}) \in \mathsf{HP}^\Gamma$. This is because (1) implies that $\mathcal{S}^* \lhd \mathsf{Lp}^{\Gamma, \Phi^+_{\mathcal{S}}}(\mathcal{S})$ or $\mathcal{S}^* \lhd \mathsf{Lp}^{\Gamma, (\Phi^+_{\mathcal{S}})^{stc}}(\mathcal{S})$. □

Next we introduce the mixed hod pair constructions.

Definition 6.37 *We say that*

$$\mathsf{mhpc} = (\mathcal{M}_\gamma, \mathcal{N}_\gamma, Y_\gamma, \Phi_\gamma, F^+_\gamma, F_\gamma, b_\gamma : \gamma \leq \delta)$$

*is the output of the **mixed hod pair construction** of $\mathbb{N}$ over $\mathcal{R}$ if the following conditions hold.*

1. *$\mathcal{M}_0 = \mathcal{J}_\omega[\mathcal{R}]$, and for all $\gamma \leq \delta$, each of $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ is either undefined or is an* hp*-indexed* lses *(see [7, Definition 3.9.2]).*
2. *For all $\gamma \leq \delta$, if $\mathcal{M}_\gamma$ is defined then $Y_\gamma = Y^{\mathcal{M}_\gamma}$ (see [7, Definition 2.3.13]).*
3. *For all $\gamma \leq \delta$, if $\mathcal{M}_\gamma$ is defined then $\Phi_\gamma = \Phi_{\mathcal{M}_\gamma}$ is the $\mathcal{E}$-certified strategy of $\mathcal{M}_\gamma$*[58].
4. *For all $\gamma \leq \delta$, if $\mathcal{N}_\gamma$ is defined and either*
   (a) *$\mathcal{N}_\gamma$ is not a reliable* hp*-indexed* lses[59] *or*
   (b) *$\mathcal{N}_\gamma$ is a reliable* hp*-indexed* lses *but for some $Q \in Y^{\mathcal{N}_\gamma}$ such that Q is meek or gentle*[60] *and for some $n < \omega$, $\rho_n(\mathcal{N}_\gamma) \leq Q$, or*
   (c) *Φ_γ is not total,*

   then all remaining objects with index $\geq \gamma$ are undefined.

   For all $\gamma \leq \eta$ for which clause 4 (the above statement) fails, $\pi_\gamma : \mathsf{core}(\mathcal{N}_\gamma) \to \mathcal{N}_\gamma$ is the uncollapse map.

[57] lses is defined in [7, Definition 2.5.4] and an lses is reliable if all of its cores exist and are iterable.
[58] See Definition 6.28.
[59] To verify that $\mathcal{N}_\gamma$ is lses, we need to verify that clause 2 of [7, Definition 2.5.4] holds.
[60] See [7, Definition 2.7.1].

5. *Suppose for some $\xi < \delta$, for all $\gamma \leq \xi$, both $\mathcal{M}_\gamma, \mathcal{N}_\gamma$ are defined. Then $\mathcal{M}_{\xi+1}$, $\mathcal{N}_{\xi+1}$, $Y_{\xi+1}$, $\Phi_{\xi+1}$, F^+_ξ, F_ξ and b_ξ are deteremined as follows.*

   (a) *Suppose $\mathcal{M}_\xi = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \in)$ is a passive hp-indexed lses, there is an extender $H^* \in \mathcal{E}$ an extender H over $\mathcal{M}_\xi$, and an ordinal $\nu < \omega\alpha$ such that $\nu < \mathrm{lh}(H^*)$ and setting*

   $$H = H^* \cap ([\nu]^\omega \times \lfloor \mathcal{M}_\xi \rfloor), \text{ and } \mathcal{N}_{\xi+1} = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \tilde{H}, \in)$$

   where $\tilde{H}$ is the amenable code of H, clause 4.a fails for $\xi + 1$. Then letting $\iota \in \mathrm{dom}(\vec{E}^{\mathcal{N}})$ be the least such that $H^* =_{def} \vec{E}^{\mathcal{N}}(\iota) \in \mathcal{E}$ has the above properties,

   $$\mathcal{N}_{\xi+1} = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \tilde{H}, \in)$$

   where $\tilde{H}$ is the amenable code of H[61]*. Assuming clause 4 fails for $\xi + 1$, the remaining objects are defined as follows.*

      i. $\mathcal{M}_{\xi+1} = \mathsf{core}(\mathcal{N}_{\xi+1})$[62],
      ii. $F^+_\xi = H^*$ *and* $F_\xi = H$,
      iii. $b_\xi = \emptyset$ *and*
      iv. $Y_{\xi+1} = \pi^{-1}_{\xi+1}(Y_\xi)$.

   (b) *Suppose $\mathcal{M}_\xi = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \in)$ is a passive hp-indexed lses*[63] *and there is an extender H over $\mathcal{M}_\xi$ such that setting*

   $$\mathcal{N}_{\xi+1} = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \tilde{H}, \in)$$

   where $\tilde{H}$ is the amenable code of H, clause 4.a fails for $\xi + 1$ and H is $\mathcal{E}$-certified as defined in Definition 6.35. Assuming clause 4 fails for $\xi + 1$, the remaining objects are defined as follows.

      i. $\mathcal{M}_{\xi+1} = \mathsf{core}(\mathcal{N}_{\xi+1})$[64],
      ii. $F^+_\xi = H^*$ *and* $F_\xi = H$,
      iii. $b_\xi = \emptyset$ *and*
      iv. $Y_{\xi+1} = \pi^{-1}_{\xi+1}(Y_\xi)$.

   (c) *Suppose $\mathcal{M}_\xi = (\mathcal{J}^{\vec{E},f}_{\omega\alpha}, \in, \vec{E}, f, Y_\xi, \in)$ is a passive hp-indexed lses, $\mathcal{M}_\xi$ is strategy-ready*[65]*, $\alpha = \beta + \gamma$ and there is $t \in \lfloor \mathcal{M}_\xi | \omega\beta \rfloor$ such that setting $w = (\mathcal{J}_\omega(t), t, \in)$, w is (f, hp)-minimal as witnessed by β. In particular, this means that we have to index the branch of t at $\omega\alpha$. and $\gamma = \mathrm{lh}(t)$. Set $b = \Phi_\xi(t)$ and*

   $$\mathcal{N}_{\xi+1} = (\mathcal{J}^{\vec{E},f^+}_{\omega\underline{+}\omega\gamma}, \in, \vec{E}, f, Y_\xi, \tilde{b}, \in)$$

   where $\tilde{b} \subseteq \omega\underline{+}\omega\gamma$ is defined by $\omega\underline{+}\omega\nu \in \tilde{b} \iff \nu \in b$. Assuming clause 4 fails for $\xi + 1$, the remaining objects are defined as follows.

      i. $\mathcal{M}_{\xi+1} = \mathsf{core}(\mathcal{N}_{\xi+1})$,
      ii. $F_\xi = F^+_\xi = \emptyset$,
      iii. $b_\xi = \tilde{b}$ *and*

[61] Here H is what is determined by H^*. For the definition of the “amenable code" see the last paragraph on page 14 of [11].
[62] Recall that $\mathsf{core}(\mathcal{M})$ is the core of $\mathcal{M}$.
[63] I.e., with no last predicate.
[64] Recall that $\mathsf{core}(\mathcal{M})$ is the core of $\mathcal{M}$.
[65] See [7, Definition 3.9.1].

iv. $Y_{\xi+1} = \pi^{-1}_{\xi+1}(Y_\xi)$.

Important Anomaly: *Suppose* $\cup Y_\xi$ *is #-lsa type and* t *is* nuvs. *Suppose* $e \in \mathcal{M}_\xi|\omega\beta$ *is such that* $\mathcal{M}_\xi|\omega\beta \vDash \mathsf{sts}_0(t, e)$[66]. *If* $e \neq b$ *then* $\mathcal{N}_{\xi+1}$ *is not an sts premouse over* $\mathcal{J}_\omega(\cup Y_\xi)$ *based on* $\cup Y_\xi$*, and so the construction must stop.*

(d) *If* $\mathcal{M}_\xi$ *doesn't satisfy clause 2a, 2b or 2c then set* $\mathcal{N}_{\xi+1} = \mathcal{J}_\omega[\mathcal{M}_\xi]$ *(this presupposes that* $Y^{\mathcal{N}_{\xi+1}} = Y_\xi$*). Assuming clause 4 fails for* $\xi + 1$*, the remaining objects are defined as follows.*

   i. $\mathcal{M}_{\xi+1} = \mathsf{core}(\mathcal{N}_{\xi+1})$[67],

   ii. $F_\xi = F^+_\xi = \emptyset$,

   iii. $b_\xi = \emptyset$,

   and $Y_{\xi+1} = \pi^{-1}_{\xi+1}(Y_\xi) \cup \{\pi^{-1}_{\xi+1}(\mathcal{M}_\xi)$ *in the case* $\mathcal{M}_{\xi+1}$ *is a hod premouse and otherwise,* $Y_{\xi+1} = \pi^{-1}_{\xi+1}(Y_\xi)$.

6. *Suppose* $\xi \leq \delta$ *is a limit ordinal and for all* $\gamma < \xi$*, both* $\mathcal{M}_\gamma$ *and* $\mathcal{N}_\gamma$ *are defined. Then* $\mathcal{M}_\xi$ *and* $\mathcal{N}_\xi$ *are determined as follows*[68]. *Set* $\mathcal{N}_\xi = lim_{\alpha\to\xi}\mathcal{M}_\alpha$. *Assuming clause 4 fails for* $\xi + 1$*, the remaining objects are defined as follows.*

   (a) $\mathcal{M}_\xi = \mathsf{core}(\mathcal{N}_\xi)$ *and*

   (b) $Y_\xi = \pi^{-1}_\xi(Y^{\mathcal{N}_\xi})$[69].

7. $\mathcal{M}_\delta = \mathcal{N}_\delta$ *and* $Y_\delta, \Phi_\delta, F^+_\delta, F_\delta$*, and* b_δ *are undefined.*

We say that the mhpc *is* ***successful*** *if for some* γ*,* $\mathcal{M}_\gamma$ *is a* Φ^+*-iterate of* $\mathcal{R}^+$. ⊣

The following is the main fact we need, which is a corollary to several lemmas established before.

Lemma 6.38 mhpc *is successful.*

Proof. The lemma follows easily from Lemma 6.36, Lemma 6.32 and (b) that appears in the proof of Lemma 6.34 (which was also established in the proof of Lemma 6.32).

To prove the lemma, we simply compare $\mathcal{R}^+$ with mhpc-construction of $\mathcal{N}$ and argue that mhpc side reaches an iterate of $\mathcal{R}^+$. As all extender used in mhpc with critical point $> \mathsf{ord}(\mathcal{R})$ have background certificates, the usual stationarity argument shows that such extenders cannot be part of a disagreement in the resulting comparison process. Lemma 6.36 shows that extenders with critical point $\mathcal{R}$ also cannot be part of a disagreement, while Lemma 6.32 shows that there cannot be a strategy disagreement. Therefore, $\mathcal{R}^+$ iterates to some model appearing on the mhpc-construction. □

Lemma 6.38 and Lemma 6.32 now imply Theorem 1.3, and this finishes our proof of Theorem 1.3.

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[66] See [7, Definition 3.8.16]. This means that e is the branch of t we must choose.

[67] Recall that $\mathsf{core}(\mathcal{M})$ is the core of $\mathcal{M}$.

[68] The rest of the objects will be defined at the next stage of the induction as in clause 4.

[69] F_ξ and b_ξ are defined at step $\xi + 1$.

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